PhD Qualify Exam. Mar. 5, 2010

(E: easy, M: moderate, D: difficult)

1. (E, 15 pts) Show that measurable sets are those which split every set (measurable or not) into pieces that are additive with respect to outer measure.

2. (E, 15 pts, 2007, 9) Show that every Lebesgue integrable function is the limit, almost everywhere, of a certain sequence of step functions.

3. (D, 20 pts, 2007, 9) $1 < p < \infty$, $f \in L^p(0, \infty)$. Define

$$F(x) = \frac{1}{x}\int_0^x f(t)dt, \quad 0 < x < \infty$$

   (a) Prove that

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p$$

   (b) Prove that the equality holds only if $f = 0$ a.e..

4. (M, 15 pts) Recall that a set is called nowhere dense if its closure has empty interior. Is there any nowhere dense set with positive Lebesgue measure ? (prove or disprove question)

5. (E, 20 pts) Let $f \in L^1(\mathbb{R}^n)$. Define (formally) the fourier transform of f by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i<x,\xi>} f(x)dx$$

   and its inverse transform by

$$f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i<x,\xi>}\hat{f}(\xi)d\xi \tag{1}$$

   Prove or disprove :
   (a) $\hat{f} \in C_0(\mathbb{R}^n)$.
   (b) The integral in equation (1) converges.

6. (E, 15 pts, 2006, 2) Let $f \in L^1(\mathbb{R})$. Define

$$F(x) = \int_{\mathbb{R}} f(t)\frac{\sin xt}{t}$$

   (a) Prove that F is differentiable on $\mathbb{R}$ and find $F'(x)$
   (b) Determine whether or not F is absolutely continuous on every compact interval of R.