國立成功大學應用數學所 數值分析 博士班資格考
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1. Definition: A sequence $\{p_n\}_{n=1}^{\infty}$ is said to be convergent to p of order α with asymptotic error constant λ if $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^{\alpha}} = \lambda$.

   (i) Let $g : [a,b] \longrightarrow [a,b]$ be a continuous function. Show that function g has at least one fixed point in (a,b), that is, there is a point $p^* \in [a,b]$ such that $g(p^*) = p^*$. 5%

   (ii) Let $p_{n+1} = g(p_n)$, with $p_0 \in (a,b)$, give a fixed point iteration. Please find a sufficient condition such that this fixed point iteration is convergent of order k, where k is a positive integer. 10%

   (iii) Show that the Newton's method is a local quadratic method (i.e. $\alpha = 2$), if the method is convergent. 5%

2. Show that if $u(x)$ is a function that interpolates $f(x)$ at $x_0, x_1, \ldots, x_{n-1}$ and $v(x)$ is a function that interpolates $f(x)$ at $x_1, x_2, \ldots, x_n$ then the function $w(x)$ given by

$$w(x) = \frac{(x_n - x)u(x) + (x - x_0)v(x)}{x_n - x_0}$$

   interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$. (10%)

3. Is it possible to use $af(x+h) + bf(x) + cf(x-h)$ with suitably chosen coefficients a, b, c to approximate $f'''(x)$? How many function values at least are required to approximate $f'''(x)$? 10%

4. Consider the initial value problem

$$\text{(I.V.P.)} \begin{cases} y' = f(t,y), & a \le t \le b, \\ y(a) = \alpha. \end{cases}$$

   (a) Show that

$$y'(t_i) = \frac{-3y(t_i) + 4y(t_{i+1}) - y(t_{i+2})}{2h} + \frac{h^2}{3}y'''(\xi_i).$$

   for some ξ_i with $t_i \le \xi_i \le t_{i+2}$. (10%)

(b) Part (a) suggests the difference method

$$w_{i+2} = 4w_{i+1} - 3w_i - 2hf(t_i, w_i), \quad \text{for } i = 0, 1, \ldots, n-2.$$

Analyze this method for consistency, stability and convergence. (10%)

5. Consider a linear system $Ax = b$, where

$$A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}.$$

(i) Choose the range of a so that A is positive definite. (5%)

(ii) Find a range of a so that the Jacobi iteration converges. (5%)

(iii) Find a range of a so that the Gauss-Seidel iteration converges. (5%)

6. Let $A = \begin{bmatrix} a & a-\varepsilon \\ 2(a+\varepsilon) & 2a \end{bmatrix}$ where $a \approx O(1)$ and ε is sufficiently small.

(a) Find A^{-1}. (5%)

(b) Choose b, δb, x and δx such that

$$Ax = b, \qquad A(x + \delta x) = b + \delta b,$$

and $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$ is small, but $\frac{\|\delta x\|_\infty}{\|x\|_\infty}$ is large. (10%)

(c) Choose b, δb, x and δx such that

$$Ax = b, \qquad A(x + \delta x) = b + \delta b,$$

and $\frac{\|\delta x\|_\infty}{\|x\|_\infty}$ is small, but $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$ is large. (10%)