GENERAL ANALYSIS

PhD Qualify Exam. Sep. 24, 2010

(E: easy, M: moderate, D: difficult)

1. (E, 15 pts, 2004, 9) Let f and $f_k, k = 1, 2, \ldots$ be measurable and finite a. e. in E, where $E \subset \mathbb{R}^n$ has finite measure. Prove that if $f_k \to f$ a. e. , then f_k converges to f in measure.

2. (E, 15 pts, 2004, 9) Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on $E \subset \mathbb{R}^n$. Prove that if $f_k \to f$ pointwise and $f_k \leq f$ for all k, then $\int_E f_k \to \int_E f$ as $k \to \infty$.

3. (E, 15 pts, 2007, 9) Show that every Lebesgue integrable function is the limit, almost everywhere, of a certain sequence of step functions.

4. (D, 20 pts, 2007, 9) $1 < p < \infty, f \in L^p(0, \infty)$. Define

$$F(x) = \frac{1}{x}\int_0^x f(t)dt, \quad 0 < x < \infty$$

   (a) Prove that

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p$$

   (b) Prove that the equality holds only if $f = 0$ a.e..

5. (E, 15 pts) Let f be a nonnegative measurable function defined on $E \subset \mathbb{R}$. Show that for any $\lambda > 0$,

$$m\{x \in E | f(x) \geq \lambda\} \leq \frac{1}{\lambda}\int_E f,$$

   where $m\{\ldots\}$ denotes the measure of that set.

6. (M, 20 pts) Let f be a bounded function on the closed, bounded interval $[a, b]$. Then f is Riemann integrable over $[a, b]$ if and only if the set of points in $[a, b]$ at which f fails to be continuous has measure zero.