PhD Qualify Exam, PDE, Sep. 21, 2001

Show all works

1.[10%] Verify that $u(x,\, t) = \dfrac{1}{2c}\displaystyle\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,\, s)\, dyds$ is the solution of

$$\begin{cases} u_{tt} + c^2 u_{xx} = f \\ u(x,0) = u_t(x,0) = 0. \end{cases} \tag{1}$$

2.[10%] Let $\phi(x)$ be a bounded continuous function for $-\infty < x < \infty$. Define the function $u(x,t) = \dfrac{1}{\sqrt{4\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{t})\, dp$. Assume that the function u is C^∞ for $-\infty < x < \infty$ and $0 < t < \infty$. (Take it for granted.) Show that $\lim_{t \to 0^+} u(x,t) = \phi(x)$.

3.(a)[10%]Let D be an open disk in R^2 of radius r and center Q. Suppose that u is harmonic in D and that $\int\int_D |u|^2\, dxdy = K < \infty$. Prove that

$$|u(Q)| \leq \frac{1}{r} \left(\frac{K}{\pi} \right)^{\frac{1}{2}}.$$

Note: You may not assume that u is continuously extendable to the closure of D.
Hint: Consider $\int\int [u - u(Q)]^2\, dxdy$.

(b)[10%] Suppose that u is harmonic on all of R^2, and that $\int\int_{R^2} |u|^2\, dxdy < \infty$. Prove that u is constant.

4.[10%] State a method of finding the Green's function for the eighth of a ball,

$$D = \{x^2 + y^2 + z^2 < a^2 : x > 0, y > 0, z > 0\}.$$

5. Let u be a solution of the wave equation in all of $R^3 \times R$. Suppose that $a > 0$ and that $u(x,0) = u_t(x,0) = 0$ for $|x| \geq a$.

(a)[10%] Show that $u(x,t) = 0$ in the double cone $|x| \leq |t| - a$ for $|t| \geq a$.

(b)[10%] Show that there is a constant $C > 0$ such that

$$\int_{R^3} u^2(x,t)\, dx \leq C, \qquad \text{for all } t > 0.$$

(Hint: Show that there is a finite energy solution of $w_{tt} - \Delta w = 0$ such that $w_t = u$.)

6. Let Ω be an open bounded subset of R^n, with smooth boundary. Consider the boundary value problem

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \gamma u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where $\gamma > 0$.

(a)[5%] Give a weak formulation of this problem.

(b)[5%] Suppose that there is a weak solution for the problem (2), which is smooth in $\overline{\Omega}$. Show that this solution satisfies (2) in the usual (strong) sense. (Recall: We say that v is a weak derivative of u if $(v, \phi) = -(u, \phi')$ for all test function ϕ.)

7. Suppose that $g \in C^1(R)$ and that $M \equiv \sup_R |g(x)| < \infty$. Consider the initial value problem

$$\begin{cases} u_t + u^2 u_x = 0 & \text{for } x \in R,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in R. \end{cases} \tag{3}$$

(a)[10%] Show that a C^1 solution $u(x,t)$ satisfies $|u(x,t)| \le M$ for as long as the solution exists.

(b)[10%] Suppose that at some $x_0 \in R$, $g(x_0)g'(x_0) < 0$. Show that a C^1 solutioin breaks down in finite time. (Hint: study the behaviour of u_x along characteristics.)