Attempt all 7 problems. Show all your work and justify all your answers.

1. (15 points) Let G be a group of order pqr, where p, q, r are primes. Show that G is solvable.

2. (15 points) Show that $\mathbb{Z}[\sqrt{-1}]$ is an Euclidean domain.

3. (15 points) Let p be a prime and $\mathbb{F}_p$ be the finite field with p elements. Find the number of monic irreducible quadratic polynomials in $\mathbb{F}_p[x]$.

4. (15 points) Let E be the splitting field over $\mathbb{Q}$ of the polynomial $x^3 - 2 \in \mathbb{Q}[x]$. Determine all subfields of E.

5. (15 points) Let M be a finite generated module over a commutative ring R with identity and $\phi : M \to M$ be an R-module homomorphism. Prove that there exists a polynomial $p(x)$ in $R[x]$ such that $p(\phi) = 0$ as an element in the R-algebra $\mathrm{Hom}_R(M, M)$ of R-module homomorphisms.

6. (15 points) Describe all the prime ideals of the polynomial ring $\mathbb{Z}[x]$ over the ring of integers.

7. (10 points) Let D be an integral domain that is a K-algebra over a field K. Show that D is a field if it is finite dimensional as a K-vector space.