Qualifying Exam in Differential Equations

September 2020

Solve all problems

(1) (M, 20 points) Let $f : [-r, r] \to \mathbb{R}$ be a C^1 function with

$$A = \frac{1}{2r} \int_{-r}^{r} f(y) dy \,.$$

Prove that

$$\int_{-r}^{r} (f(x) - A)^2 \, dx \le (2r)^2 \int_{-r}^{r} (f'(x))^2 \, dx \,.$$

(2) (E, 20 points) Consider the transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + y\dfrac{\partial u}{\partial x} = 0 \,, & (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \,, \\ u(0, x, y) = u_0(x, y) \,, & \end{cases}$$

where $u_0(x, y)$ is a continuous function with $u_0(x, y) = 0$ for $|x| + |y| \ge 1$.
a. Solve the equation.
b. Prove that for fixed $t_0 > 0$, $y_0 \in \mathbb{R}$,

$$\lim_{x \to \infty} u(t_0, x, y_0) = 0 \,.$$

(3) (E, 20 points) Let Ω be a open, bounded, connected subset of $\mathbb{R}^2$ and u, v be harmonic functions with $u \ge v$ on $\partial\Omega$.
a. Prove that either $u > v$ on Ω or $u = v$ on $\overline{\Omega}$.
b. Give an example to show that the conclusion in part a. fail if Ω is open, connected, but not bounded in $\mathbb{R}^2$.

(4) (E, Old problem, 20 points) Find the solution of the problem:

$$\begin{cases} u_t = u_{xx} \,, & \\ u(0, x) = 0 \,, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \,. \\ u(t, 0) = h(t), \quad h(0) = 0 \,, & \end{cases}$$

(5) (E, Old problem, 20 points) Using the energy method to prove the uniqueness of the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} + hu = F(x,t)\,, \quad x \in \mathbb{R}, t > 0 \\ \lim_{x\to\pm\infty} u(x,t) = \lim_{x\to\pm\infty} u_t(x,t) = \lim_{x\to\pm\infty} u_x(x,t) = 0\,, \quad t > 0 \\ \int_{-\infty}^{\infty} u_t^2 + c^2 u_x^2 + hu^2 dx < \infty\,, \quad t > 0 \\ u(0,x) = f(x)\,, u_t(0,x) = g(x) \quad x \in \mathbb{R}\,, \end{cases}$$

where c and h are positive constants.