PhD Qualify Exam

General Analysis

Mar, 2021

1. (15 pts) For $f : [0,1] \to \mathbb{R}$, let $E \subset \{x|f'(x)\text{exists}\}$. If $m(E) = 0$, show that $m(f(E)) = 0$

2. (15 pts) Let (X, μ) be a measure space, $\{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty}$ are two sequences of measurable functions on X. Assume that g_n converges in measure to some g, and that for any fixed $\varepsilon > 0$, $\lim_{n\to\infty} \mu(\{x \in X \,|\, |f_n(x) - g_n(x)| > \varepsilon\}) = 0$. Show that f_n converges in measure to g.

3. (15 pts) Suppose f_n is a sequence of measurable functions on $[0,1]$ with $\|f_n\|_{L^2([0,1])} \le M$ for all $n \in \mathbb{N}$ and for some $M > 0$ and $f_n \to 0$ a.e. on $[0,1]$. Prove that $\|f_n\|_{L^1([0,1])} \to 0$ as $n \to \infty$.

4. (15 pts) Prove the following generalization of Hölder's inequality. If $\sum_{i=1}^k 1/p_i = 1/r$, $p_i, r \ge 1$, then
$$\|f_1 \cdots f_k\|_{L^r} \le \|f_1\|_{L^{p_1}} \cdots \|f_k\|_{L^{p_k}}.$$

5. (20 pts) Let $f \in L([0,\infty))$ and $a > 0$. Show that
$$\int_0^\infty \int_0^\infty \sin(ax) f(y) e^{-xy} dy dx = a \int_0^\infty \frac{f(y)}{a^2 + y^2} dy.$$

6. (20 pts, 2014) Let $\phi(x) \ge 0$ be a bounded measurable function in $\mathbb{R}^n$. $\phi(x) = 0$ for $|x| \ge 1$ and $\int \phi dx = 1$. for $\varepsilon > 0$, let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$.

   (a) If $f \in L^1(\mathbb{R}^n)$, show that $\lim_{\varepsilon\to 0}(f * \phi_\varepsilon)(x) = f(x)$ a.e..

   (b) If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, show that $\|(f * \phi_\varepsilon)(x) - f(x)\|_p \to 0$ as $\varepsilon \to 0$.

   Recall that $(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy$.