Ph.D Qualify Exam **Analysis** **2020 Fall**

E: Easy; M: Moderate; D: Difficult

1(E, 15%, 2019, Spring). A sequence $\{f_n\}$ of Lebesgue measurable functions is called Cauchy sequence in measure if given $\varepsilon > 0$ there is N such that

$$\text{Leb}\left(\{x \mid |f_n(x) - f_m(x)| \geq \varepsilon\}\right) < \varepsilon$$

for all $m, n > N$, where Leb$(\cdot)$ represents the Lebesgue measure. (a) Write down the definition of the convergence in measure. (b) Prove that $\{f_n\}$ converges in measure.

2(M, 15%, 2018, Spring). Suppose that $f_k \to f$ in $L^3(\mathbb{R}^n)$, $g_k \to g$ a.e., and there exists $M > 0$ such that $\|g_k\|_{L^\infty(\mathbb{R}^n)} < M$ for all k. Prove that $f_k g_k \to fg$ in $L^3(\mathbb{R}^n)$.

3(E, 15%, 2019, Fall). Let $k(x, y)$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying that

$$\int_{\mathbb{R}^n} |k(x,y)|dy \leq C \text{ for a.e. } x \text{ and } \int_{\mathbb{R}^n} |k(x,y)|dx \leq C \text{ for a.e. } y,$$

where $C > 0$ is a universal constant. Prove that

$$(Tf)(x) := \int_{\mathbb{R}^n} k(x,y) f(y) dy$$

is a bounded operator on $L^p(\mathbb{R}^n)$ with $\|Tf\|_p \leq C\|f\|_p$ for $1 \leq p \leq \infty$.

4(E, 15%, 2018, Fall). Let $\{f_k\}$ and f be Lebesgue measurable functions on a measurable set $E \subset \mathbb{R}^n$, where Leb$(E) < \infty$. Prove that

$$f_k \to f \text{ in measure if and only if } \int_E \frac{|f_k(x) - f(x)|}{1 + |f_k(x) - f(x)|} dx \to 0 \text{ as } k \to \infty.$$

5(E, 15%). The total variation function of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_f(x) = \sup\left\{\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \ : \ n \in \mathbb{N}, \ -\infty < x_0 < \cdots < x_n = x\right\}, \ x \in \mathbb{R}.$$

If $\lim_{x \to \infty} T_f(x)$ exists and is finite, prove that the function $T_f + f$ is increasing.

6(E, 10%). Let $(X, \mathcal{M})$ be a measurable space. Suppose that μ and ν are measures on $(X, \mathcal{M})$ with $\nu \ll \mu$. Define a new measure λ by $\lambda = 2\mu + \nu$. Denote the Radon-Nikodym derivative of ν with respect to λ by f. Express the Radon-Nikodym derivative of ν with respect to μ in terms of f.

7(M, 15%). Suppose f_k, $f \in L^1(\mathbb{R}^n)$ and $f_k \to f$ a.e. Prove or disprove that $\int_{\mathbb{R}^n} |f_k(x)|dx \to \int_{\mathbb{R}^n} |f(x)|dx$ implies $\int_{\mathbb{R}^n} |f_k(x) - f(x)|dx \to 0$.