PhD Qualify Exam in Numerical Analysis

October 18, 2017

1. (2010 Fall, Easy) Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by $x_{n+1} = g(x_n)$, where $g(x)$ is a function defined on a given interval $[a, b] \subset \mathbb{R}$.

   (a) (5%) Give a sufficient condition for $g(x)$ so that it has a fixed point in $[a, b]$.

   (b) (5%) Give a sufficient condition for $g(x)$ so that $\{x_n\}_{n=0}^{\infty}$ is convergent of order k, where k is a positive integer.

   (c) (5%) Show that the Newton's iteration is locally quadratically convergent provided the iteration converges to a simple root.

2. (2017 Spring, Average) (10%) Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular and that we have solutions to linear systems $Ax = b$ and $Ay = g$ where b, $g \in \mathbb{R}^n$ are given. Show how to solve the system

$$\begin{bmatrix} A & g \\ h^T & \alpha \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} b \\ \beta \end{bmatrix}$$

   in $O(n)$ flops, where $\alpha, \beta \in \mathbb{R}$ and $h \in \mathbb{R}^n$ are given and the enlarged matrix $\begin{bmatrix} A & g \\ h^T & \alpha \end{bmatrix}$ is nonsingular.

3. (2006 Fall, Easy) 10% Give an example to illustrate that $(a+b)c \neq ac+bc$ may happen in a real calculator.

4. (Average) Consider the 2×2 matrix $A = \begin{bmatrix} 1 & \rho \\ -\rho & 1 \end{bmatrix}$

   (a) (7%) Under what conditions will Gauss-Seidal iteration converge with this matrix?

   (b) (8%) For what range of ϖ will the SOR method converge?

   (c) (5%) What is the optimal choice for the parameter ϖ?

5. (Average) We learn in calculus that $e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$.

   (a) (10%) Assume that there are constants $K_1, K_2, \ldots$ such that

   $$e = (1+h)^{\frac{1}{h}} + K_1 h + K_2 h^2 + K_3 h^3 + \cdots .$$

   Use extrapolation on the approximations to produce an $O(h^3)$ approximation to e when h is sufficient small.

   (b) (5%) Do you think that the assumption in part 6.(a) is reasonable?

6. (2010 Fall, Easy) (15%) Is it possible to use $af(x+h)+bf(x)+cf(x-h)$ with suitably chosen coefficients a, b, c to approximate $f'''(x)$? How many function values at least are required to approximate $f'''(x)$?

7. (2011 Spring, Challenged) (15%) Consider the initial-value problem

   $$y' = f(t, y), \quad \text{for} \quad a \le t \le b \quad \text{with} \quad y(a) = \alpha$$

   Let

   $$w_0 = \alpha$$
   $$w_{i+1} = w_i + h\phi(t_i, w_i, h) \quad \text{for} \quad i > 0,$$

   and

   $$\widetilde{w}_0 = \alpha$$
   $$\widetilde{w}_{i+1} = \widetilde{w}_i + h\widetilde{\phi}(t_i, \widetilde{w}_i, h) \quad \text{for} \quad i > 0,$$

   give two one-step methods for approximating solution of $y(t)$ with local truncation error $\tau_{i+1}(h) = O(h^n)$ and $\widetilde{\tau}_{i+1}(h) = O(h^{n+1})$, respectively, i.e.,

   $$y(t_{i+1}) = y(t_i) + h\phi(t_i, y(t_i), h) + O(h^{n+1})$$

   and

   $$y(t_{i+1}) = y(t_i) + h\widetilde{\phi}(t_i, y(t_i), h) + O(h^{n+2}).$$

   Please use these two methods to construct an adaptive step-size control method for the considering initial-value problem.