PhD Qualify Exam, Analysis, Oct. 9, 2014

Show all works

E: Easy, M:Moderate, D: Difficult

1.[10%] (Sept. 2008) Evaluate the integral $\int_{-\infty}^{\infty} e^{-x^2} \cos(xt)\, dx$.

2.[20%] [E] Let C be the Cantor set and $\varphi(x)$ be the Cantor function. **(a)** State the definintion of the Cantor set. **(b)** Show that the Lebesque measure of C is zero, that is $m(C) = 0$. **(c)** Show that C is uncountable. **(d)** State the definintion of the Cantor function $\varphi(x)$. **(e)** Is $\varphi(x)$ continuous? uniformly continuous? absolutely continuous? Prove your assertion. **(f)** Show that $C + C = [0, 2]$. **(g)** Using the definition to find the Hausdorff measure of C.

3.[20%] (Sept. 2008) Let $\mathbf{T}(x, y) = (e^x \cos y - 1, e^x \sin y) = (u, v)$ be a transformation: $R^2 \to R^2$, and f be a continuous function on R^2 with compact support. Let $J_{\mathbf{T}}$ be the Jacobian of $\mathbf{T}$. **(a)** Show that there are functions g_1 and g_2 from R^2 into R^1 such that $\mathbf{T}(x, y) = \mathbf{G_2} \circ \mathbf{G_1}(x, y)$, where $\mathbf{G_1}(x, y) = (g_1(x, y), y)$ and $\mathbf{G_2}(z, w) = (z, g_2(z, w))$. **(b)** Show that, for Riemann integral, $\int_{R^2} f(u, v)\, dudv = \int_{R^2} f(\mathbf{T}(x, y)) \big| J_{\mathbf{T}}(x, y) \big|\, dxdy$. Use the result in part (a) to give a direct proof. **(c)** Under what conditions, does the formula in part (b) hold for Lebesgue integral?

4.[10%](Sept. 2004) Assume that $p > 0$ and $\int_E |f - f_k|^p dx \to 0$ as $k \to \infty$. Show that $\{f_k\}_{k=1}^{\infty}$ converges in measure on E to f.

5.[10%](Feb. 2007) Let $1 < p < \infty$, $f \in L^p(0, \infty)$, $F(x) = \dfrac{1}{x} \int_0^x f(t)\, dt$, and $0 < x < \infty$. **(a)** Prove that $\|F\|_p \leq \dfrac{p}{p-1} \|f\|_p$. **(b)** Prove that the equality holds only if $f = 0$ *a.e.* **(c)** What can you say about $p = 1$ and $p = \infty$?

6.[20%](Feb. 2000) Let $\mathcal{M}$ be the collection of Lebesgue measurable subsets of R. μ be the Lebesgue measure on $(R, \mathcal{M})$, and μ_0 be the counting measure on $(R, \mathcal{M})$. Define ν on $(R, \mathcal{M})$ by $\nu(E) = \mu_0(E \cap \{0\}) - \mu(E \cap [0, 1]) + \int_E \dfrac{1}{1 + x^2}\, dx$. $(E \in \mathcal{M})$ **(a)** Find a Hahn decomposition of R for measure ν. **(b)** Find the Jordan decomposition of ν. **(c)** Find the Lebesgue decomposition of $|\nu|$ with respect to μ. **(d)** Compute the Radon-Nikodym derivative of the absolutely continuous part of $|\nu|$ with respect to μ.

7.[10%] **(a)** State the Riesz representation theorem of the dual of $L^p(E)$.

(b) State the Radon-Nikodym Theorem.

(c) State the Riesz representation theorem of the dual of $L^p(X, \mu)$.

(d) State the Riesz representation theorem of the dual of $C(X, \mu)$.