Qualified Examination: Partial Differentiation Equation

June, 2012 (E: easy, M: moderate, D: difficult)

1. Let u be a nonegative harmonic function in a ball $B_R(0)$. Show that for $|x| < R$,

$$R^{n-2}(R-|x|)u(0)/(R+|x|)^{n-1} \leq u(x) \leq R^{n-2}(R-|x|)u(0)/(R-|x|)^{n-1}.$$

(20 points(M))

2. Solve

$$(y+u)u_x + yu_y = x - y$$

subject to the initial condition $u(x,1) = 1 + x$. (20 points (M))

3. Use the Fourier transform to find an explicit formula for u, where u satisfies

$$-\Delta u + u = f \quad \text{in} \quad \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$. (20 points(M))

4. Suppose $u \in C^2(\bar{\Omega})$ be a solution of

$$\begin{cases} \Delta u(x) = f(x) & \text{in} \quad \Omega, \\ u(x) = g(x) & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is the unit ball in $\mathbb{R}^n$ with center at the origin and $f, g \in C(\bar{\Omega})$. Prove that there exists a constant C such that

$$\max_{\Omega} |u(x)| \leq C(\max_{\partial\Omega} |g(x)| + \max_{\Omega} |f(x)|). \quad \text{(20 points(D))}$$

5. Consider the problem

$$\begin{cases} u_t - 4u_{xx} = x^7 t^5 & 0 < x < 1,\ t > 0, \\ u_x(0,t) = u_x(1,t) = 3t & t \geq 0, \\ u(x,0) = \cos \pi x & 0 \leq x \leq 1. \end{cases}$$

Prove that the solution is unique. (20 points(E))