PhD Qualify Exam, PDE, Mar. 02, 2012

Show all works

E: easy, M: moderate, D: difficult, O: old exam

1.(O) Solve the wave equation for infinite vibrating string $u_{tt} = c^2(x)u_{xx}$, where $c(x) = \begin{cases} c_1, & x < 0. \\ c_2, & x > 0. \end{cases}$ Let a wave $u(x,t) = f(x - c_1 t)$ come in from the left. Thus the initial conditions are $u(x,0) = f(x)$ and $u_t(x,0) = -c_1 f'(x)$. Assume that $u(x,t)$ and $u_x(x,t)$ are continuous everywhere. Also give an interpretation for the solution you find. [10%]

2.(O) Let $u, v \in C^1(\overline{\Omega})$ be conjugate harmonic functions, i.e., $u_x = v_y$ and $u_y = -v_x$, in a simply connected domain Ω with C^1 boundary in R^2. Show that on the boundary curve $\partial\Omega$, $\dfrac{du}{dn} = \dfrac{dv}{ds}$, $\dfrac{dv}{dn} = -\dfrac{du}{ds}$, where $\dfrac{d}{dn}$ denotes differentiation in the direction of the outer normal and $\dfrac{d}{ds}$ differentiation in the counter-clockwise tangential direction. Show that these relations can be used to reduce the Neumann problem for u to the Dirichlet problem for v. [10%]

3.(O) Let $\Omega \subset R^n$ be open. Show that if there exists a function $u \in C^2(\overline{\Omega})$ vanishing on $\partial\Omega$ for which the quotient $\dfrac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$ reaches its infimum λ, then u is an eigenfunction for the eigenvalue λ, so that $\Delta u + \lambda u = 0$ in Ω. Let us call them λ_1 and u_1. How do we find λ_2 and u_2? Give an example of eigenvalue problem and find $\lim_{n\to\infty} \lambda_n$ in your example. [10%]

4.(a)(O) Let u be a solution of the wave equation in all of $R^3 \times R$. Suppose that $a > 0$ and that $u(\mathbf{x}, 0) = u_t(\mathbf{x}, 0) = 0$ for $|\mathbf{x}| \geq a$. Show that $u(\mathbf{x}, t) = 0$ in the double cone $|\mathbf{x}| \leq |t| - a$ for $|t| \geq a$. [10%]

(b)(E) Answer the same question for the wave equation in $R^2 \times R$. [5%]

(c)(M) Find the fundamental solution (or Riemann function, or Green's function, or source function) $S(\mathbf{x}, t)$ for the wave equation and the solution u of $\begin{cases} u_{tt} - \Delta u = f(\mathbf{x}, t), \ \mathbf{x} \in R^3, \ t \in R, \\ u(\mathbf{x}, 0) = g(\mathbf{x}), \ u_t(\mathbf{x}, 0) = h(\mathbf{x}), \ \mathbf{x} \in R^3, \end{cases}$ in terms of S, f, g, and h. [10%]

(d)(M) Use Fourier transform to find the solution $u(\mathbf{x}, t)$ in terms of $\widehat{f}(\xi, t)$, $\widehat{g}(\xi)$, and $\widehat{h}(\xi)$. [5%]

5. In this problem set we always assume that the Neumann function H exists.

(a)(E) Analogous to the Green's function G, please state the definition of the Neumann function $H(x, y)$ for the operator $-\Delta$ and the domain $D \in R^2$ at the point $\mathbf{x}_0 \in D$. [5%]

(b)(M) Find the solution of the problem $\begin{cases} \Delta u = f, \ \text{in } D \\ \frac{\partial u}{\partial n} = h, \ \text{on} \partial D. \end{cases}$ (Hint: Use the Green's Identities.) [5%]

(c)(D) Solve the Neumann problem in the half-plane $\begin{cases} \Delta u = f, \ \text{in} \{y > 0\}, \\ \frac{\partial u}{\partial n} = h, \ \text{on} \{y = 0\}, \end{cases}$ with u bounded at ∞. (Hint: Consider the problem satisfied by $v = \dfrac{\partial u}{\partial y}$.) [10%]

6.(M) Find the solution for the diffusion equation on the half-line: $\begin{cases} u_t - u_{xx} = f(x,t), & \\ u(x,0) = g(x), & x > 0,\ t > 0, \\ u(0,t) = h(t), & \end{cases}$

where $g(0) = h(0) = 0$. [10%]

7.(D) Find a traveling wave solution of $u_t + u_{xxx} + 6uu_x = 0 \quad (-\infty < x < \infty)$, that is, $u(x,t) = f(x-ct)$. Also assume that $f(x), f'(x), f''(x)$ tend to 0 as x tends to $\pm\infty$. (Hing: $f(x) = \frac{1}{2}c\,\text{sech}^2\left[\frac{1}{2}\sqrt{c}(x - x_0)\right]$, where x_0 is an integration constant and c the wave speed.) [10%]

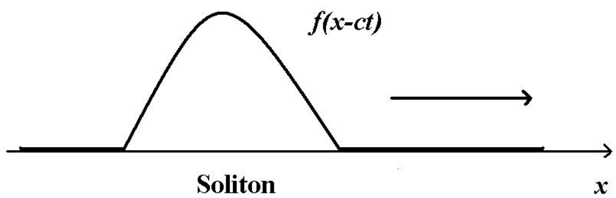

Figure 1: