國立成功大學應用數學所 數值分析 博士班資格考
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1. (100上, Average) Consider a linear boundary value problem with Dirichlet boundary conditions:

$$u''(x) = f(x) \quad \text{for } 0 < x < 1$$
$$u(0) = \alpha, \quad u(1) = \beta.$$

We attempt to compute a grid function consisting of values U_0, U_1, $\ldots$, U_m, U_{m+1} where U_j is our approximation to the solution $u(x_j)$. Here $x_j = jh$ and $h = 1/(m+1)$ is the mesh width, the distance between grid points. From the boundary conditions we know that $U_0 = \alpha$ and $U_{m+1} = \beta$ and so we have m unknown values U_1, $\ldots$, U_m to compute. We solve this problem with a centered difference scheme,

$$\frac{1}{h^2}(U_{j-1} - 2U_j + U_{j+1}) = f(x_j) \quad \text{for } j = 1, 2, \ldots, m.$$

The problem can be written in the form $AU = F$ where U is the vector of unknowns $U = [U_1, U_2, \ldots, U_m]^T$ and

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}, \qquad F = \begin{bmatrix} f(x_1) - \alpha/h^2 \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{bmatrix}. \tag{1}$$

(a) (5 %) Compute the local truncation error of this scheme.

(b) (10 %) Find the eigenvalues of A.

(c) (5 %) Show that $||A^{-1}||_2$ is uniformly bounded as $h \to 0$ and the numerical scheme is stable in the **2-norm**.

(d) (5 %) Show that the numerical scheme is convergent in the **2-norm**.

2. (Difficult)

   (a) (10 %) Determine the Green's functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with Dirichlet conditions at $x = 0$ and $x = 1$, i.e, find the function $G(x, \bar{x})$ solving
   $$u''(x) = \delta(x - \bar{x}), \quad u(0) = 0, \quad u(1) = 0$$

   (b) (10 %) Find the general formulas for the elements of the inverse of the matrix A in Problem 1. Write out A^{-1} for the case $h = 0.25$. Note that finding A^{-1} is equivalent to solving linear systems
   $$A\,v = e_j,$$
   where e_j is a unit vector with the value 1 as its jth element and all other elements equal to 0, and the linear system is a discrete version of the problem
   $$v''(x) = h\delta(x - x_j)$$
   with homogeneous boundary conditions.

   (c) (5 %) Show that $||A^{-1}||_\infty$ is uniformly bounded as $h \to 0$ and the numerical scheme is stable in the **max-norm**.

3. (Average)

   Consider the heat equation with Dirichlet boundary conditions:
   $$\begin{aligned} &u_t = u_{xx} \quad \text{for } 0 < x < 1,\ 0 < t < T, \\ &u(0,t) = 0, \quad \text{for } 0 < t < T, \\ &u(1,t) = 0, \quad \text{for } 0 < t < T. \end{aligned}$$

   We attempt to solve the problem using a finite difference scheme on a discrete with grid points (x_i, t_n) where $x_i = ih$, $t_n = nk$. Here $h = 1/(m+1)$ is the mesh spacing on the x-axis and k is the time step. Let $U_i^n \approx u(x_i, t_n)$ represent the numerical solution at grid point (x_i, t_n).

   The finite difference is
   $$\begin{aligned} U_i^{n+1} &= U_i^n + \frac{k}{h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n), . \quad \text{for i=1,\ldots, m,} \\ U_0^n &= 0, \quad U_{m+1}^n = 0. \end{aligned}$$

(a) (10 %) Determine the order of accuracy of this method (in both space and time).

(b) (10 %) Suppose we take $k = \lambda h^2$ for some fixed $\lambda > 0$ and refine the grid. Show that this method is stable for $0 < \lambda \leq 1/2$ and hence convergent.

(c) (5 %) Based on the CFL Condition, a numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as k and h go to zero.

What is the domain of dependence of the heat equation? Does this explicit scheme satisfy the CFL condition?

4. (97上, Average) (15 %) Prove that if a nonsingular matrix A has an LU-factorization in which L is a lower triangular matrix with unit diagonals, i.e., $L(i,i) = 1$ for all i , then the LU-factorization is unique.

5. (Easy) Let $A \in \mathbb{C}^{m \times n}$ be full rank. Verify the following statements.

(a) (5 %) If $m > n$, then $X = (A^H A)^{-1} A^H$ gives a Moore-Penrose pseudo-inverse of matrix A.

(b) (5 %) If $m < n$, then $X = A^H (A A^H)^{-1}$ gives a Moore-Penrose pseudo-inverse of matrix A.

Reference

The following are some definitions related to the above questions.

1. Matrix norms

   - $||A||_\infty = \max_{1\le i\le m} \sum_{j=1}^{m} |a_{ij}|$.
   - $||A||_1 = \max_{1\le j\le m} \sum_{i=1}^{m} |a_{ij}|$.
   - $||A||_2 = \sqrt{\rho(A^T A)}$, where $\rho(B)$ denotes the spectral radius of the matrix B (the maximum modulus of an eigenvalue).

2. Grid-functions norms For a grid function $e = (e_0, \cdots, e_N)$ on a uniform grid with spacing h,

   - $||e||_\infty = \max_{0\le i\le N} |e_i|$.
   - $||e||_1 = h\sum_{i=0}^{N} |e_i|$.
   - $||e||_2 = \left(h\sum_{i=0}^{N} |e_i|^2\right)^{1/2}$.

3. The Moore-Penrose pseudo-inverse is a matrix X of the same dimensions as A^H satisfying four conditions:

   (i) $AXA = A$, (ii) $XAX = X$, (iii) $(AX)^H = AX$, (iv) $(XA)^H = XA$.