GENERAL ANALYSIS
PhD Qualify Exam. Mar. 2, 2012

(E: easy, M: moderate, D: difficult)

1. (E, 15 pts, 2012, 3) Show that measurable sets are those which split every set (measurable or not) into pieces that are additive with respect to outer measure.

2. (E, 15 pts, 2011, 6) Let $\{E_n\}_{n=1}^{\infty}$ be a descending sequence of measurable sets. If there exists j such that the measure of E_j, $m(E_j) < \infty$, then it holds $m(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$. Show also that the existence of such j is essential.

3. (M, 20 pts, 2010, 9) Let f be a bounded function on the closed, bounded interval $[a, b]$. Then f is Riemann integrable over $[a, b]$ if and only if the set of points in $[a, b]$ at which f fails to be continuous has measure zero.

4. (E, 15 pts) Show that the sequence $f_n(x) = \frac{1}{n}\sum_{1\leq k\leq n} \sin^2 kx$ converges in measure in $(-\pi, \pi)$.

5. (E, 15 pts) Let $a \in \mathbb{R} \setminus \{0, 1\}$. We denote the decimal representation of $x \in (0, 1)$ by $x = \sum_{k\geq 1} \epsilon_k(x) \cdot 10^{-k}$. Ddefine the function

$$f(x) = \begin{cases} a, & \epsilon_n(x) \neq 0, \forall n \\ 1, & \text{if the first } \epsilon_n(x) \neq 0 \text{ has even index} \\ 0, & \text{otherwise} \end{cases}$$

   Show that f is measurable and find $\int_0^1 f$.

6. (E, 20 pts) Prove or disprove

   (a) If f is a strictly increasing continuous function with $f'(x) = 0$ a.e. , then f is a constant function.

   (b) If f is an absolutely continuous function with $f'(x) = 0$ a.e. , then f is a constant function.