FINAL FOR NUMBER THEORY

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No credit will be given for an answer without reasoning.

1. [10%] Prove that $\sqrt{2}+\sqrt{3}$ is a root of $x^4 - 10x^2 + 1 = 0$, and hence establish that it is irrational.

2. [10%] Suppose that p is an odd rational prime. Show that

$$\left(\frac{3}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{3}; \\ -1, & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

where $\left(\frac{a}{p}\right)$ denotes the Legendre symbol.

3. [10%] Recall that the function μ is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } a^2 | n \text{ for some rational integer } a > 1; \\ (-1)^r, & \text{if } n = p_1 p_2 \cdots p_r \text{ where } p_1, \dots, p_r \text{ are distinct rational primes.} \end{cases}$$

Find a positive rational integer n such that $\mu(n) + \mu(n+1) + \mu(n+2) = 3$.

4. [10%] Show that the primitive solutions of $x^2 + y^2 = z^2$ with y even are $x = r^2 - s^2$, $y = 2rs$, $z = r^2 + s^2$, where r and s are arbitrary integers of opposite parity with $r > s > 0$ and $\gcd(r, s) = 1$.

5. [10%] Let $\{a_i\}$ and $\{b_i\}$ be increasing sequences of real numbers. We say that a_i is asymptotic to b_i, write $a_i \sim b_i$, if and only if $\lim_{i\to\infty} a_i/b_i = 1$. Prove that $a_i \sim b_i$ implies $\ln a_i \sim \ln b_i$, but the converse is not true.

6. [10%] Let $a_0, a_1, a_2, \dots$ be a sequence of rational integers, all positive except perhaps a_0. Recall that we define the sequence of rational integers $\{k_n\}$ by $k_{-2} = 1$, $k_{-1} = 0$ and $k_i = a_i k_{i-1} + k_{i-2}$ for $i \geq 0$. Show that $k_n/k_{n-1} = \langle a_n, a_{n-1}, \dots, a_2, a_1\rangle$ for $n \geq 1$.

7. [10%] Let k and r be positive rational integers with $k > 1$ and $r > 1$. Prove that there is a rational prime whose digital representation to base r has exactly k digits.

8. [10%] Show that 3 is a prime in $\mathbb{Q}(\sqrt{-1})$, but not a prime in $\mathbb{Q}(\sqrt{6})$.

9. [10%] The rational prime 13 can be factored in two ways in $\mathbb{Q}(\sqrt{-3})$,

$$13 = \frac{7+\sqrt{-3}}{2} \cdot \frac{7+\sqrt{-3}}{2} = \big(1+2\sqrt{-3}\big)\big(1-2\sqrt{-3}\big).$$

Explain why this does not conflict with the fact that $\mathbb{Q}(\sqrt{-3})$ has the unique factorization property.

10. [10%] Let $p(n)$ denote the number of partitions of n. Compute $p(9)$.