本試題是否可以使用計算機：□可使用 ，☑不可使用 （請命題老師勾選）

1. (10 points) Let $g_n : I = [0,1] \to \mathbb{R}$ be defined by $g_n(x) = \frac{1}{nx+1}$. Determine whether g_n converge on $I = [0,1]$ and, if it converges, determine whether the convergence is uniform.

2. (10 points) Let $f : I = [a,b] \to \mathbb{R}$ be (Riemann) integrable on I and assume that f is continuous at $c \in (a,b)$. Prove that $\lim_{r\to 0} \frac{1}{2r}\int_{c-r}^{c+r} f(x)dx = f(c)$.

3. (10 points) Let D be the rectangle in $\mathbb{R}\times\mathbb{R}$ given by $D = \{(x,t) | a \le x \le b, c \le t \le d\}$. Let f and its partial derivative f_t be continuous functions defined on D, and F be a function defined on $[c,d]$ given by $F(t) = \int_a^b f(x,t)dx$. Prove that F has a derivative on $[c,d]$ and $F'(t) = \int_a^b f_t(x,t)dx$.

4. Let $\sup S$ denotes the supremum (or the least upper bound) of S, and $\inf S$ denotes the infimum (or the greatest lower bound) of S.
   (a) (6 points) Let $I = (a,b)$ be an open interval in $\mathbb{R}$, and let f and g be continuous functions defined on I. Prove that the function $h : I \to \mathbb{R}$ defined by $h(x) = \sup\{f(x), g(x)\}$ is continuous on I.
   (b) (6 points) Let X and Y be non-empty sets and let $f : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Prove that $\sup_y \inf_x f(x,y) \le \inf_x \sup_y f(x,y)$.

5. Let $F : \mathbb{R}^5 \to \mathbb{R}^2$ be defined by $F(u,v,w,x,y) = (uy + vx + w + x^2, uvw + x + y + 1)$, and note that $F(2,1,0,-1,0) = (0,0)$.
   (a) (6 points) Show that we can solve $F(u,v,w,x,y) = (0,0)$ for (x,y) in terms of (u,v,w) near $(2,1,0)$.
   (b) (6 points) If $(x,y) = \phi(u,v,w)$ is the solution of the preceding part, show that $D\phi(2,1,0)$ is given by the matrix $-\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & -1 & -3 \\ 0 & 1 & -3 \end{pmatrix}$.

6. (10 points) Define a sequence of real numbers (x_n) by $x_0 = 1$, and $x_{n+1} = \frac{1}{2+x_n}$, for $n \ge 0$. Show that (x_n) converges and compute its limit. [**Hint:** Use the contraction principle.]

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $\lim_{x\to\infty} f(x) = 0$.
   (a) (10 points) Show that there exists a sequence $x_n \to \infty$ with $\lim_{n\to\infty} f'(x_n) = 0$.
   (b) (6 points) Show that it is not necessarily true that $f'(x)$ is bounded.

8. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each n, so that

$$|f_n'(x)| \le 1, \quad \text{for all } x \in \mathbb{R}, n = 1, 2, \cdots.$$

   (a) (6 points) Prove that the set $\{f_n\}$ is uniformly equicontinuous on $\mathbb{R}$. [**Hint:** A set $\mathscr{F}$ of functions on K to $\mathbb{R}^n$ is said to be uniformly equicontinuous on K if, for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, y \in K$ and $\|x - y\| < \delta(\varepsilon)$ and $f \in \mathscr{F}$, then $\|f(x) - f(y)\| < \varepsilon$.]
   (b) (6 points) For each n, let $\tilde{f}_n(x) = f_n(x) - f_n(0)$. Prove that $\{\tilde{f}_n\}$ is uniformly bounded on any closed interval $[a,b] \subset \mathbb{R}$.
   (c) (8 points) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is such that for each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = g(x)$. Prove that g is continuous.