Show all your work. Explanation is required for each problem.

1. [10%] Compute the surface integral $\iint_S x^2\,dS$ where S is the unit sphere $x^2+y^2+z^2=1$.

2. A set $A \subset \mathbb{R}^n$ is said to be *dense* in a set $B \subset \mathbb{R}^n$ if $A \subset B$ and $B \subset \mathrm{cl}(A)$ (the closure of A).
   (i) [6%] If A is dense in $\mathbb{R}^n$ and $U \subset \mathbb{R}^n$ is open, prove that $A \cap U$ is dense in U.
   (ii) [6%] Give an example of a set $A \subset \mathbb{R}^n$ and a closed set $V \subset \mathbb{R}^n$ such that A is dense in $\mathbb{R}^n$ but $A \cap V$ is not dense in V.

3. (i) [6%] Give an example of a bounded function $f\colon [0,1] \to \mathbb{R}$ such that $|f|$ is Riemann-integrable on $[0,1]$ but f is not Riemann-integrable on $[0,1]$.
   (ii) [6%] Give an example of a function which is bounded and continuous but not uniformly continuous.

4. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) := \begin{cases} 0, & \text{if } (x,y) = (0,0); \\ \frac{x^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0). \end{cases}$$

   (i) [6%] Prove that f is continuous.
   (ii) [6%] Prove that f is not differentiable at $(0,0)$.

5. For $n = 1,2,3,\ldots$ and $x \in \mathbb{R}$, we define

$$f_n(x) := \frac{x}{1+nx^2}.$$

   (i) [6%] Prove that the sequence $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a differentiable function f.
   (ii) [6%] Prove that $f' \neq \lim_{n\to\infty} f_n'$ at some point in $\mathbb{R}$.

6. Define $\ln(x) := \int_1^x \frac{1}{t}\,dt$ for $x > 0$.
   (i) [6%] Prove (from above definition) that $\ln(ab) = \ln(a) + \ln(b)$ for any $a, b > 0$.
   (ii) [6%] Prove that $\lim_{x\to 0} \ln(x) = -\infty$.
   (iii) [6%] Prove that $\ln(x)$ has a differentiable inverse function (denoted by $\exp(x)$) and prove that $\frac{d}{dx}\exp(x) = \exp(x)$.

7. Let A be a subset of $\mathbb{R}$. We define $\lambda(A) \in \mathbb{R} \cup \{\infty\}$ as follows. First, if A is an open interval (a,b), then we define $\lambda(A) := b - a$. Second, if A is an open set, we know that A is a union of countable (including finite) disjoint open intervals: $\bigcup_{k=1}^{\infty}(a_k, b_k)$ (or $\bigcup_{k=1}^{n}(a_k, b_k)$). Then we define $\lambda(A) := \sum_{k=1}^{\infty}(b_k - a_k)$ (or $\sum_{k=1}^{n}(b_k - a_k)$). Finally, if A is any subset of $\mathbb{R}$, we define

$$\lambda(A) := \inf\{\, \lambda(X) \mid A \subset X,\ X \subset \mathbb{R} \text{ and } X \text{ is open} \,\}.$$

   (i) [6%] Show that $\lambda([a,b]) = b - a$.
   (ii) [6%] If $A \subset B \subset \mathbb{R}$, prove that $\lambda(A) \le \lambda(B)$.
   (iii) [6%] Suppose we know that $\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \lambda(A_k)$ for subsets $A_1, A_2, A_3, \ldots$ of $\mathbb{R}$. Compute $\lambda(\mathbb{Q})$.
   (iv) [6%] Give an example of subsets $A_1, A_2, A_3, \ldots$ of $\mathbb{R}$ such that $A_1 \supset A_2 \supset A_3 \supset \cdots$ and $\lambda(\lim_{k\to\infty} A_k) \neq \lim_{k\to\infty} \lambda(A_k)$.