1. Assume $a_n > 0$, $s_n = a_1 + a_2 + \cdots + a_n$, and $\sum a_n$ diverges.
(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges. (10%)
(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges. (10%)
(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges. (10%)
(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \text{ and } \sum \frac{a_n}{1+n^2a_n} \text{ ?}$$

(10%)

2. Show that the improper integral $\int_0^\infty \frac{\sin x}{x}\,dx$ exists and find its value. (20%)

3. Let $\sum a_n x^n$ be a power series. Assume that $\sum a_n x^n$ converges at $x_0 \neq 0$. Show that the radius of convergence of $\sum a_n x^n$ is at least $|x_0|$. (10%)

4. Let $f(x,y)$ be defined as $f(0,0) = 0$ and

$$f(x,y) = x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4+y^2)^2}$$

if $(x,y) \neq (0,0)$.
(a) Prove that $f(x,y)$ is continuous for all $(x,y) \in \mathbf{R}^2$. (Hint: show $4x^4y^2 \leq (x^4+y^2)^2$) (10%)
(b) Prove that the restriction of f to each straight line through $(0,0)$ has a local minimum at $(0,0)$. (10%)
(c) Is $(0,0)$ a local minimum for f ? (10%)