Advanced Calculus (Master program)

1. (10%) Given the sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ defined by
$$(a_1, b_1) = (1, 1), \qquad a_{n+1} = a_n + b_n, \qquad b_{n+1} = 2a_n + b_n$$
Please compute the limit
$$\lim_{n\to\infty} \frac{b_n}{a_n}.$$
You need to give a proof of the convergence.

2. (10%) Discuss the convergence of the p-series
$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad p > 0$$
without applying any theorem.

3. (10%) Discuss the convergence of the following two integrals
$$\text{(a)} \quad \int_0^{\infty} \frac{\sin x}{x} dx, \qquad \text{(b)} \quad \int_0^{\infty} \left|\frac{\sin x}{x}\right| dx$$

4. (10%) Prove or disprove there is a continuous map taking $[0,1]$ onto $(0,1)$. How's the case taking $[0,\infty)$ to $(-\infty,-1]\cup[1,\infty)$?

5. (10%) Given a function $f : [0,1] \mapsto [0,1]$. Prove that f has a fixed point, i.e., there exists $x \in [0,1]$ such that $f(x) = x$.

6. (10%) Given a function f defined by the power series
$$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{n^2}, \qquad x \in [0,1]$$
Show that f is a continuous on $[0,1]$.

7. (10%) Given a function $f : [a,b] \mapsto [0,\infty)$. Is it true that if $\int_a^b f(x)dx = 0$ then $f = 0$?

8. (15%) Choose proper α, β and γ such that
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-(\alpha x^2 + 2\beta xy + \gamma y^2)]dxdy = 1.$$

9. (15%) For what value of k is the function r^k integrable in $\mathbf{R}^n$, where
$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$