Part I. 高等微積分

Prove or disprove the following statements

1. (10)$\varnothing \neq S \subset \mathbb{R}$, S is connected $\Longrightarrow$ S is an interval.
2. (10)$x \in \mathbb{R} \setminus \{2m\pi : m \in \mathbb{Z}\} \Longrightarrow |\sum_{k=1}^{n} e^{ikx}| \leq \frac{1}{|\sin \frac{x}{2}|}$
3. (10)If $\sum a_n$ is convergent and $\{b_n\}$ is a convergent sequence, then $\sum a_n b_n$ is convergent.
4. (10)
$$y = \begin{cases} \sin \frac{1}{x}, & x \in (0,1] \\ 0. & x \in [-1,0] \end{cases}$$
The set $S = \{(x,y) | x \in [-1,1]\} \subset \mathbb{R}^2$ is arcwise connected.
5. (10)
$$\lim_{m\to\infty} [\lim_{n\to\infty} \cos^{7n}(m!\sqrt{7}\pi)] = -1.$$
6. (10) The sequence $\{x^n\}_{n=1}^{\infty}$ converges in $[0,1]$ and the convergence is not uniform.
7. (20) Let S be a subset of a metric space. Then S is compact if and only if every sequence in S contains a convergent subsequence in S.
8. (20) Let V be a subspace of the Euclidean space $(\mathbb{R}^{2009}, d)$. Given $x \in \mathbb{R}^{2008}$, there exists a unique $y \in V$ with $d(x,y) = \inf_{z\in V} d(x,z)$.

Part II. 線性代數

In the following, $\mathbb{R}$ denote the set of real numbers.

1. (10 points total, 2pts each) For each of the following statement, answer "T" for *true* or "F" for *false* . No need to justify your answer.

   In the following, m and n are both positive integers.

   (a) Let $\phi : \mathbb{R}^m \mapsto \mathbb{R}^n$ be a linear map. Then the kernel $\phi^{-1} = \{v \in \mathbb{R}^m | \phi(v) = 0\}$ forms a sub-vector space of $\mathbb{R}^m$.

   (b) For any $m \times n$ matrix A and $n \times m$ matrix B, $\det AB = \det BA$.

   (c) If A is a square matrix and B is similar to A, then A and B have the same eigenvectors.

   (d) Let A be a 4×4 matrix with characteristic polynomial

   $$x(x-1)(x-2)^2 \text{ and minimal polynomial } x(x-1)(x-2).$$

   Then A can be diagonalized.

   (e) If $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by $T(x, y) = (2x + y + 1, x - y)$, then there exists a 2×2 real matrix A such that $T(x, y) = A \cdot (x, y)^t$, where $(x, y)^t$ denotes the transpose of (x, y).

Show all your work for the rest of problems.

2. (10pts) Let E be the real vector space spanned by $v_1 = (1, 1, 0, 0)$, $v_2 = (0, 2, -1, 1)$. Let $v = (0, 2, 0, 4)$. Find the orthogonal projection (with respect to the standard inner product) of v into E.

3. (10pts)

   (a) (5pts) Let A be an $m \times n$ matrix over $\mathbb{R}$. If $A^t Ax = 0$ for some $x \in \mathbb{R}^n$, show that $Ax = 0$.

   (b) (5pts) Use part (a) to show that if the columns of A are linearly independent, then $A^t A$ is invertible.

4. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

   (a) (10pts) Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

   (b) (10pts) Find the matrix $B = A^{33} - 5A^{32} + 3A^{31} + 9A^{30} + A^3 - 5A^2 + 4A + 4I$, where I denotes the 3×3 identity matrix.