90 academic year

Part I.

(1) Evaluate the following integrals (15%)

(a) $\int_0^\infty (\frac{\sin x}{x})^2\, dx$. (Hint: $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$).

(b) $\int_0^\infty \frac{\cos x}{1+x^2}\, dx$.

(2) For sequence $\{a_n\}$, $a_1 \le a_2 \le \cdots \le M$ for some $M \in \mathbb{R}$ (15%)

(a) Prove $\lim\limits_{n\to\infty} a_n = \sup\{a_n\}$, where $\sup\{a_n\}$ denotes the supremum of $\{a_n\}$.

(b) Use (a) to prove $\lim\limits_{n\to\infty} (1+\frac{1}{n})^n$ exists.

(c) Prove the $\lim\limits_{x\to\infty} (1+\frac{1}{x})^x$ exists.

(3) Let $\{f_n\}$ be a sequence of continuous functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ such that $\{f_n\}$ converges to f on D, and let $\{x_n\}$ be a sequence in D which converges to $x \in D$. Prove or disprove $\{f_n(x_n)\}$ converges to $f(x)$. (10%)

(4) Prove $C = \{x : x = \sum\limits_{n=1}^{\infty} \frac{a_n}{3^n},\ a_n = 0 \text{ or } 2\}$ is uncountable. (10%)

Part II.

(5)

(a) Give an example of a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ without any eigenvalues. (3%)

(b) Give an example of a linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ without any eigenvalues. (3%)

(6) Find the characteristic polynomial, minimal polynomial, eigenvalues and eigenvectors of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (2x + y, y - z, 2y + 4z)$$

for all $(x, y, z) \in \mathbb{R}^3$. Is it possible to diagonalize T? Why? (6%)

(7) Let U_1 and U_2 be vector subspaces of a finite dimensional vector space V. Show that there is an isomorphism $T\colon V \to V$ such that T maps U_1 isomorphically onto U_2 if and only if $\dim U_1 = \dim U_2$. (8%)

(8) Let V be a finite dimensional vector space over $\mathbb{R}$ and $\langle \cdot, \cdot \rangle$ a symmetric bilinear form on V with $\langle x, x \rangle \ge 0$ for all $x \in V$. Let $\varphi\colon V \to V$ be a linear transformation. Prove that the following statements are equivalent:

(a) For all $x, y \in V$, $\langle \varphi(x), y \rangle = -\langle x, \varphi(y) \rangle$.

(b) The matrix A of φ relative to some orthogonal basis of V is anti-symmetric (i.e., $A^T = -A$). (10%)

(9) Let V be an n-dimensional vector space over some field F, and let $T\colon V \to V$ be a linear transformation. Let $v \in V$ be such that $\{v, T(v), T^2(v), \cdots\}$ spans V. Suppose that $\dim V = n$. Show that the vectors, $v, T(v), T^2(v), \ldots, T^{n-1}(v)$ form a basis of V. (10%)

(10) Let V be a vector space and $n \in \mathbb{N}$. Suppose that $f\colon V \to V$ is a linear transformation with $f^n(v) = 0$ for all $v \in V$. Let $g\colon V \to V$ be an arbitrary linear transformation and set $T = f \circ g = g \circ f$. Show that there is some $m \in \mathbb{N}$ such that $T^m(v) = 0$ for all $v \in V$. (10%)