87 academic year

1. Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, u^t be the transpose of u, $A = (u^t)u$ and $\lambda = \|u\|^2$.
Prove that A is similar to $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. (10%)

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be linear, rank $T = 2$ and A be the representation matrix of T w.r.t.
the standard basis of $\mathbb{R}^3$.
   (i) Prove that ran $T^* = (\ker T)^\perp$. (7%)
   (ii) List all the possible canonical forms of A. (7%)
   (iii) Find the sufficient and necessary conditions on T so that A is unitarily similar to
   a diagonal matrix. Justify your answer ! (10%)

3. Let $C[a, b]$ be the collection of all real–valued functions which are continuous on $[a, b]$.
Define $\langle f, g\rangle = \int_a^b f(x)g(x)dx$ for $f, g \in C[a, b]$.
   (i) Prove that $\langle \cdot, \cdot\rangle$ is an inner product on $C[a, b]$. (7%)
   (ii) Let $\|f\| = (\langle f, f\rangle)^{\frac{1}{2}}$. Prove or disprove that if $f_n \in C[a, b]$ for $n = 1, 2, 3, \dots$ and $\|f_n - f\| \to 0$, then $f \in C[a, b]$. (7%)

4. Let $I \subseteq \mathbb{R}$ and $f_n \to f$ uniformly on I.
Prove or disprove the following statements:
   (i) if each f_n is bounded on I, then f is bounded on I. (7%)
   (ii) if $g_n \to g$ uniformly on I, then $f_n g_n \to fg$ uniformly on I. (7%)

5. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $p > 0$. Show that $f(\lambda x) = \lambda^p f(x)$ for every real λ and every x in $\mathbb{R}^n$, if and only if $x \cdot \nabla f(x) = pf(x)$ for all x in $\mathbb{R}^n$. (10%)

6. Let $f(x, y) = \int_0^\infty \frac{1}{(1+x^2t^2)(1+y^2t^2)}dt$ if $(x, y) \neq (0, 0)$.
   (i) Show that $f(x, y) = \frac{\pi}{2} \cdot \frac{1}{x+y}$. (8%)
   (ii) Evaluate the $\int_0^1 dy \int_0^1 f(x, y)dx$ to derive the formula:
$$\int_0^\infty \frac{(\tan^{-1} x)^2}{x^2} dx = \pi \log 2.$$
(10%)

7. Suppose $S = \{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1\}$, $p = (1, 0, 0)$ and $\int_S d\sigma$ means the surface integral on S. Prove
$$\frac{1}{2\pi^2} \int_S (p \cdot u)u d\sigma(u) = p.$$
(10%)