國立成功大學 108 學年度「碩士班」研究生甄試入學考試

【基礎數學】: Part I. 線性代數

Linear Algebra

In the following, $F^{m\times n}$ denotes the class of all $m \times n$ matrices with entries in the field F, where $F = \mathbb{R}$ or $\mathbb{C}$. Vectors in F^n will be regarded as column vectors. We say a matrix $A \in \mathbb{R}^{n\times n}$ is symmetry if $A^T = A$, a matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if $A^* = \overline{A}^T = A$. A matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if $x^T Ax > 0$ for all nonzero $x \in \mathbb{R}^n$.

(1) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $U = \{X \in \mathbb{R}^{2\times 2} : AX = XA\}$, find the dimension of U. (20 points)

(2) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta \in [0, 2\pi]$.
a. Show that $A^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$ for all $n \in \mathbb{N}$. (10 points)
b. Calculate A^{-n} for all $n \in \mathbb{N}$. (10 points)

(3) Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix.
a. Show that all eigenvalues of A are real. (10 points)
b. If (λ_1, y_1) and (λ_2, y_2) are two eigenpairs of A with $\lambda_1 \neq \lambda_2$, show that $\langle y_1, y_2\rangle = 0$. (10 points)

(4) Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be symmetric positive definite matrix, show that
a. $a_{ii} > 0$ for all $1 \le i \le n$. (10 points)
b. $a_{ii}a_{jj} > a_{ij}^2$ for all $i \neq j$. (10 points)

(5) Let $w_i \in \mathbb{R}$, $1 \le i \le 4$ and $A = \begin{bmatrix} w_1w_1 & w_1w_2 & w_1w_3 & w_1w_4 \\ w_2w_1 & w_2w_2 & w_2w_3 & w_2w_4 \\ w_3w_1 & w_3w_2 & w_3w_3 & w_3w_4 \\ w_4w_1 & w_4w_2 & w_4w_3 & w_4w_4 \end{bmatrix}$ with $w_1^2 + w_2^2 + w_3^2 + w_4^2 = 1$.
a. Find all eigenvalues of A and its algebraic multiplicity. (10 points)
b. Calculate $\det(I_4 - 2A)$. (10 points)

【基礎數學】：Part II. 高等微積分

1. (10 points) Evaluate

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dydx$$

by interchanging limits of integrations.

2. (10 points) A function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if for every $s < t$ and $\lambda \in [0, 1]$,

$$f(\lambda s + (1-\lambda)t) \le \lambda f(s) + (1-\lambda)f(t).$$

Prove that if a convex function is differentiable, its derivative f' is an increasing function:

$$s \le t \Rightarrow f'(s) \le f'(t).$$

Note: Here we do *NOT* assume the existence of f''. It is useful to consider the fact that for $s < t$, the graph of f over $[s, t]$ always lies below the line through $(s, f(s))$ and $(t, f(t))$.

3. For a subset E of a metric space (X, d), let E^o be the set of interior points of E. Prove that
   (a) (10 points) E^o is open.
   (b) (10 points) If $G \subset E$ and G is open, then $G \subset E^o$

   Note: $x \in E$ is an *interior point* if there exists $r > 0$ so that

$$B_r(x) = \{y \in X \mid d(x, y) < r\} \subset E.$$

4. (10 points) Given a sequence of Riemann integrable functions $\{f_n\}$ on $[a, b]$ and assume that f_n converges uniformly to a function f, it is known that f is also Riemann integrable on $[a, b]$. With this fact, prove that

$$\int_a^b f\, dx = \lim_{n\to\infty} \int_a^b f_n\, dx.$$

5. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Define the *operator norm* of A by

$$\|A\| := \sup_{|\mathbf{x}|_n \leq 1} \{|A\mathbf{x}|_m\}.$$

Here, $|\cdot|_n$ and $|\cdot|_m$ are the usual Euclidean lengths in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

(a) (10 points) Prove that

$$\|A\| = \sup_{|\mathbf{x}|_n = 1} \{|A\mathbf{x}|_m\}.$$

(b) (10 points) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$\|A\| = ?$

6. (10 points) Prove that the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x, y, z) = (e^{x^3+2x}, e^{y-z} + \cos z, e^z)$$

is locally invertible. That is, for every $P \in \mathbb{R}^3$, there exists a neighborhood U of P so that $f|_U : U \to f(U)$ is invertible.

7. Consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$ on $[0, 1]$ given by

$$f_n(x) = \frac{\sin^4(nx)}{\sin^2(nx) + (1 - nx)^2}.$$

(a) (10 points) Prove that $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded. That is, there exists $M > 0$ so that $|f_n(x)| \leq M$ for all n, and all $x \in [0, 1]$.

(b) (10 points) Use Arzela-Ascoli Theorem to prove that $\{f_n\}_{n=1}^{\infty}$ is *NOT* equicontinuous.