國立成功大學 107 學年度「碩士班」研究生甄試入學考試

【基礎數學】：Part I. 線性代數

In the following, $F^{m\times n}$ denotes the class of all $m \times n$ matrices with entries in the field F, where $F = \mathbb{R}$ or $\mathbb{C}$. Vectors in F^n will be regarded as column vectors. $F^{m\times n}$ and F^n are vector spaces over F in the canonical way.

Justify all your answers for the problems below.

1. Let $W \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations $AX = 0$, where $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$. Find a basis for W. (15 points)

2. Let L be the line $y = mx$ in $\mathbb{R}^2$, where $m \in \mathbb{R}$. Find the matrix $A \in \mathbb{R}^{2\times 2}$ so that $x \mapsto Ax$ is the orthogonal projection onto L. (15 points)

3. Compute $\det(M)$, where M is the following $n \times n$ tridiagonal matrix:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}. \text{ (15 points)}$$

4. Suppose $n \geq m$. Let $v_1, \ldots, v_m$ be linearly independent vectors in $\mathbb{C}^n$, and $K_1, \ldots, K_m$ be linear subspaces of $\mathbb{C}^n$. Let A be the subspace of $\mathbb{C}^{n\times n}$ containing all matrices M such that $Mv_j \in K_j$ for $j = 1, 2, \ldots, m$. Find $\dim(A)$ (in terms of $n, \dim(K_1), \ldots, \dim(K_m)$). (20 points)

5. Let $P = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Give the necessary and sufficient condition on a, b such that $\lim_{n\to\infty} P^n$ exists. (20 points)

6. Find a nonsingular $Q \in \mathbb{C}^{3\times 3}$ such that $A = QJQ^{-1}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, and J is the Jordan form of A. (15 points)

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【基礎數學】: Part II. 高等微積分

1. (25 points) Given a set X,
   (a) (5 points) State the definition for a function $d : X \times X \to \mathbb{R}$ to be a *metric*.
   (b) (5 points) State the definition for a subset $E \subset X$ to be *open* with respect to d.
   (c) (5 points) State the definition for a subset $E \subset X$ to be *closed* with respect to d.
   (d) (5 points) Let $X = \mathbb{R}$ and $E = \{x\} \subset \mathbb{R}$ for some $x \in \mathbb{R}$. Prove that E is closed with respect to the metric $d(x, y) = |x - y|$.
   (e) (5 points) Define a metric d on $\mathbb{R}$ so that the subset E in part (d) is open. Explain your answer.

2. (15 points) Assume *Bolzano-Weirstrass Theorem* on $\mathbb{R}$ with the usual Euclidean metric:

   Any bounded sequence has a convergent subsequence,

   prove the same theorem for $\mathbb{R}^2$, also with standard Euclidean metric

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

3. (15 points) Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$,
   (a) (5 points) State the definition for f to be *differentiable* at $\mathbf{p} \in \mathbb{R}^n$.
   (b) (10 points) Show, *directly* (ie. without using big theorem) from the definition in part (a), that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (xy + x, y^2 + y)$$

   is differentiable at $(0, 0)$.

4. (15 points) Consider the sequence of functions $\{f_n\}$ defined on $[0, 1]$ given by

$$f_n(x) = \log nx^2(1 - x^3)^n.$$

   (a) (5 points) Find $f(x) := \lim_{n\to+\infty} f_n(x)$.
   (b) (10 points) Prove that f_n are not uniformly convergent.

5. (15 points) If a function $\phi : \mathbb{R} \to \mathbb{R}$ satisfies

$$|\phi(x) - \phi(y)| \leq \frac{1}{2}|x - y|,$$

   prove that
   (a) (5 points) ϕ is continuous (with respect to the usual metric $d(x, y) = |x - y|$).
   (b) (10 points) $\phi(x) = x$ for some $x \in \mathbb{R}$.

6. (15 points) Consider $\mathcal{C}([0,1])$, the space of continuous complex valued functions on $[0,1]$.

   (a) (5 points) State the *Stone-Weirstrass Theorem* on this space.

   (b) (10 points) Prove that if $f \in \mathcal{C}([0,1])$ satisfies the fact that

$$\int_0^1 x^n f(x)dx = 0 \ \ \forall n \in \mathbb{N},$$

   then $f = 0$. (Hint: first show that $\int_0^1 (f(x))^2 \ dx = 0$ using part (a).)