國立成功大學 106 學年度「碩士班」研究生甄試入學考試

【基礎數學】: Part II. 高等微積分

1. (20%)

   (a) (10%) State the L'Hôpital's Rule

   (b) (10%) State the Cauchy-Schwarz inequality for sequences

2. (10%) Let $E = \{(x, y, z) \mid 4 \leq x^2 + y^2 + z^2 \leq 9\}$. Compute

$$\iiint_E x^2 + y^2 + z^2 \, dV$$

3. (15%) Prove or disprove the following statements

   (a) (5%) Every sequence in $\mathbb{R}$ contains a monotone (nondecreasing or nonincreasing) subsequence.

   (b) (5%) Let $\{f_n\}$ be a sequence of continuous functions on $[0, 1]$ and converge pointwisely to a continuous function f on $\mathbb{Q} \cap [0, 1]$. Then $\{f_n\}$ converges pointwisely to f on $[0, 1]$.

   (c) (5%) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then there exists $\delta > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

   for $x \in (-\delta, \delta)$.

4. (35%) Let (X, d) be a complete metric space and $f : \mathbb{R}^n \to X$ be a function with

$$d\big(f(x), f(y)\big) \leq C\|x - y\|$$

   where $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$ and $C > 0$ is a constant.

   (a) (5%) Prove that f is uniformly continuous.

   (b) (10%) Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}^n$. Prove that there exists a convergent subsequence of $\{f(x_n)\}$ in X.

   (c) (10%) If $g : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with $\|Dg(x)\| < \frac{1}{2}$ for all $x \in \mathbb{R}^n$, prove that there exists $x_0 \in \mathbb{R}^n$ such that $g(x_0) = x_0$.

   (d) (10%) Define $h : \mathbb{R}^n \to \mathbb{R}^n$ by $h(x) = x + g(x)$. Prove that h is one to one on $\mathbb{R}^n$.

5. (20%) Let $k(x) : \mathbb{R}^n \to \mathbb{R}$ be a smooth and nonnegative function with integral 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with $|f(x)| \leq e^{-\|x\|}$ for every $x \in \mathbb{R}^n$. For $\epsilon > 0$, define $k_\epsilon(x) = \epsilon^{-n} k(\frac{x}{\epsilon})$ and $f_\epsilon(x) := \int_{\mathbb{R}^n} k_\epsilon(y) f(x - y) \, dy$. Prove that $f_\epsilon \to f$ uniformly as $\epsilon \to 0$.

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【基礎數學】：Part I. 線性代數

Note: $\mathbb{R}$ denotes the field of real numbers, and n denotes a positive integer.

1. (10%) Is there a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,-2,0) = (1,1)$, $T(3,-5,1) = (2,3)$, and $T(-1,3,1) = (3,0)$? *Justify your answer.*

2. (15%) Let V be the vector space of all polynomials of degree at most n with real coefficients. For $i = 0, 1, \ldots, n$, let $p_i(x) = x^i + x^{i+1} + \cdots + x^n \in V$. Show that $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ is a basis for V.

3. (20%) Let A be an $n \times n$ real matrix such that $A^2 = A$. Show that the trace of A is equal to the rank of A. Is A similar over $\mathbb{R}$ to a diagonal matrix? *Justify your answer.*

4. (20%) Let T be a linear operator on a finite-dimensional vector space such that $\mathrm{rank}(T^2) = \mathrm{rank}(T)$. Show that $\mathrm{N}(T) \cap \mathrm{R}(T) = \{0\}$. (Here $\mathrm{N}(T)$ and $\mathrm{R}(T)$ are the null space and the range of T respectively.)

5. (15%) Let V be the vector space of all polynomials of degree at most 3 with real coefficients. Let D be the linear operator on V defined by $D(p) = p'$ for $p \in V$. Find the Jordan form of D.

6. (20%) Let T and U be linear operators on an n-dimensional vector space V. Suppose that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is a basis for V for some $v \in V$, and that $TU = UT$. Show that $U = p(T)$ for some polynomial p.