advanced calculus

1. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $a_1, \cdots, a_n$ are real numbers and n is a positive integer. Given that $|f(x)| \le |\sin x|$ for all real x, prove that

$$|a_1 + 2a_2 + \cdots + na_n| \le 1.$$

(15 points)

2. If $a_0, a_1, \cdots, a_n$ are real numbers satisfying

$$\tfrac{a_0}{1} + \tfrac{a_1}{2} \cdots + \tfrac{a_n}{n+1} = 0,$$

show that the equation $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ has at least one real zero. (15 points)

3. Define a function f by

$$f(x) = \begin{cases} x \cdot \sin(1/x), & x \ne 0, \\ 0, & x = 0. \end{cases}$$

Prove or disprove that f is Lipschitz continuous on $\mathbb{R}$. (15 points)
(A function f is Lipschitz continuous on $\mathbb{R}$ if there is a constant L such that

$$|f(x) - f(y)| \le L|x - y| \quad \text{for} \quad \text{all} \quad x, y \in \mathbb{R}.)$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differential and $|f'(x)| \le c < 1$ on $\mathbb{R}$. Assume that $f(0) > 0$. Show that there exists a solution $\tilde{x} > 0$ to the equation $f(\tilde{x}) = \tilde{x}$. (15 points)

5. Let $f \in C[0,1]$ and $f(0) = 0$. Show that

$$\lim_{n \to \infty} \int_0^1 f(x^n)\,dx = 0.$$

(20 points)

6. Consider the pair of equations

$$\begin{cases} x^2 + y^2 + (z-1)^2 = 4, \\ -x^2 - y + z^2 = 1. \end{cases}$$

(a) Is there a curve of intersection through $(0, 0, -1)$? (10 points)
(b) Is there a curve of intersection through $(\sqrt{3}, 0, 2)$? (10 points)

Linear Algebra

Read carefully the definitions and terminology given in the following section **before** you work on any of the problems. Work out all of the problems and show details of your works.

Definitions and Terminology. In what follows, we fix F for a field and V for a vector space over F. The field of real numbers is denoted by $\mathbb{R}$ and the field of complex numbers is denoted by $\mathbb{C}$. The set of all positive integers is denoted by $\mathbb{N}$. Let $L(V, V)$ be the set of all linear transformations from V to V, and let $M(n, F)$ be the set of all $n \times n$ matrices, where n is a positive integer.

We call a linear transformation $f \in L(V, V)$ *nilpotent* if there exists some $n \in \mathbb{N}$ such that $f^n(v) = 0$ for all $v \in V$, and we call f *cyclic* if there exists some $v \in V$ such that V is spanned by v, $f(v)$, $f^2(v), \ldots, f^{n-1}(v)$. A subspace W of V is called *cyclic with respect to* f if it is spanned by w_0, $f(w_0)$, $f^2(w_0), \ldots, f^{n-1}(w_0)$ for some $w_0 \in W$.

[24%] 1. (a) Let $\mathscr{C}^\infty(\mathbb{R})$ denote the vector space of all real valued functions on $\mathbb{R}$ which has derivatives of every order. Consider the differential operator (a linear transformation) $D : \mathscr{C}^\infty(\mathbb{R}) \to \mathscr{C}^\infty(\mathbb{R})$ given by

$$D(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y,$$

where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$. Show that $e^{\lambda x}$ lies in the kernel of D if and only if λ is a root of the polynomial

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(b) Find two linearly independent solutions to the homogeneous differential equation

$$y'' - 5y' + 4y = 0.$$

(You have to show that they are indeed linearly independent.)

[36%] 2. (a) Let $v_1, v_2, \ldots, v_n$ be a basis of V and let $T \in L(V, V)$. Prove that T is nilpotent if and only if there exist positive integers $r_1, r_2, \ldots, r_n$ such that $T^{r_j}(v_j) = 0$ for $j = 1, 2, \ldots, n$.

(b) Let $T \in L(V, V)$ be nilpotent, and W a one-dimensional subspace of V. Show that W is cyclic with respect to T if and only if W lies in the kernel of T.

(c) Let $T \in L(V, V)$ be cyclic. Show that for $U \in L(V, V)$, $TU = UT$ if and only if $U = g(T)$ for some polynomial $g(x) \in F[x]$.

[20%] 3. Let $T \in L(V, V)$. Prove that there exists a nonzero linear transformation $S \in L(V, V)$ such that $TS = 0$ if and only if there exists a nonzero vector $v \in V$ such that $T(v) = 0$.

[20%] 4. Let $n \geq 2$ and let $A, B \in M(n, \mathbb{C})$ be such that $AB = BA$. Show that A and B has a common eigenvector.

Work out all problems.

1. (6 points) Let E_1 and E_2 be two subsets of $\mathbb{R}^n$ such that $E_1 \subset E_2$ and $E_2 \setminus E_1$ is countable. Show that

$$|E_1|_e = |E_2|_e.$$

2. (6 points) Find a set $E \subset \mathbb{R}$ with outer measure zero and a function $f : E \to \mathbb{R}$ such that f is continuous on E and $f(E) = [0,1]$.

3. (8 points) Give an example of a sequence of measurable functions $\{f_k\}$ defined on a measurable set$E \subset \mathbb{R}^n$ such that the following strict inequalities hold:

$$\int_E \liminf_{k\to\infty} f_k \, dx < \liminf_{k\to\infty} \int_E f_k \, dx < \limsup_{k\to\infty} \int_E f_k \, dx < \int_E \limsup_{k\to\infty} f_k \, dx.$$

4. (10 points) Suppose E is a Lebesgue measurable subset of $\mathbb{R}$ with $|E| < \infty$. Prove that

$$|E| = \sup\{|K| : K \subset E \text{ and } K \text{ is compact}\}.$$

5. (10 points) Let $f : E \to \mathbb{R} \bigcup \{\pm\infty\}$ be a nonnegative measurable function such that $\int_E f < \infty$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable subset $E_1 \subset E$ with $|E_1| < \delta$ we have $\int_{E_1} f < \varepsilon$.

6. (10 points) Let f_k be a sequence of nonnegative measurable functions defined on E. If $\lim_{k\to\infty} f_k = f$ and $f_k \le f$ a.e. on E. Show that $\lim_{k\to\infty} \int_E f_k = \int_E f$.

(1) (6 points) Show that any closed subspace of a compact space is compact.

(2) (6 points) Let $\alpha\colon \mathbb{R} \to \mathbb{R}^3$ be the smooth curve given by

$$\alpha(s) = \frac{1}{\sqrt{2}}\left(s,\ \sqrt{1+s^2},\ \log(s+\sqrt{1+s^2})\right).$$

Here log means natural logarithm. Show that α is unit speed, and compute its curvature and torsion. You may use the following calculus formulas:

$$\frac{d}{ds}\sqrt{1+s^2} = \frac{s}{\sqrt{1+s^2}}, \qquad \frac{d}{ds}\log(s+\sqrt{1+s^2}) = \frac{1}{\sqrt{1+s^2}}.$$

(3) (6 points) Let $S \subset \mathbb{R}^3$ be the set

$$S = \{\, (x,y,z) \mid y^2 = xz \text{ and } y > 0 \,\}.$$

Show that S is a regular surface.

(4) (6 points) Describe the region of the unit sphere covered by the image of the Gauss map of the paraboloid of revolution $z = x^2 + y^2$.

(5) Consider the Enneper's surface

$$\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$$

and show that

(a) (2 points) The coefficients of the first fundamental form are

$$E = G = (1+u^2+v^2)^2, \quad F = 0.$$

(b) (2 points) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

(c) (4 points) The principal curvatures are

$$k_1 = \frac{2}{(1+u^2+v^2)^2}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}.$$

(d) (4 points) The lines of curvature are the coordinate curves.

(e) (4 points) The asymptotic curves are $u + v =$ constant, $u - v =$ constant.

(6) (10 points) Let $S \subset \mathbb{R}^3$ be a regular, compact, orientable surface which is not homeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero.

Algebra

Work out all of the problems and show details of your works.

Notation. $\mathbb{Z}$ is the ring of integers. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are the fields of rational numbers, real numbers, and complex numbers, respectively.

[8%] 1. Let G be a finite group acting on a set S. Let $s \in S$ be fixed and denote $G_s = \{g \in G \mid g \cdot s = s\}$, the isotropy group of s. Show that the order of the orbit $Gs = \{g \cdot s \mid g \in G\}$ is equal to the index $[G : G_s]$ of the subgroup G_s in G.

[12%] 2. (a) Let E be a finite extension field of the field F, and K a finite extension field of E. Show that $[K : F] = [K : E] \cdot [E : F]$.

(b) Let E be an extension field of the field F, and let $\alpha, \beta \in E$ be algebraic over F. Are $\alpha + \beta$ and $\alpha\beta$ algebraic over F? Why or why not?

(c) Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

[8%] 3. Let G be a group and let $\operatorname{End} G$ be the set of all homomorphisms from G to G with additive and multiplicative binary operations on $\operatorname{End} G$ defined as follows:

$$(f + g)(a) = f(a)g(a), \text{ and } f \cdot g(a) = f(g(a)),$$

for any $f, g \in \operatorname{End} G$ and $a \in G$. Show that $\operatorname{End} G$ is a ring if and only if G is abelian.

[12%] 4. (a) Let R a ring and $a \in R$. Let I be the ideal generated by a. What is a typical element in I?

(b) Show that in $\mathbb{Q}[x]$ every ideal is generated by a single element.

[10%] 5. Let $R = \{a + b\boldsymbol{i} \mid a, b \in \mathbb{R}\}$, a subring of $\mathbb{C}$, where $\boldsymbol{i}$ is the element satisfing $\boldsymbol{i}^2 = -1$. Let M be an $\mathbb{R}$-module. Prove that M is an R-module if and only if there exists an $\mathbb{R}$-homomorphism $\varphi : M \to M$ such that $\varphi^2(x) = -x$ for all $x \in M$.