注意事項：作答時請務必在所屬答案卷上作答並標明題號。 98.05.11

1. (10 points) Prove that the geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

converges uniformly on any closed interval $[a, b] \subset (-1, 1)$.

2. (8 points) Prove that the volume bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $4\pi abc/3$.

3. (8 points) Suppose that $\{E_\alpha\}_{\alpha \in A}$ is a collection of connected sets in a Euclidean space R^n such that $\bigcap_{\alpha \in A} E_\alpha \neq \phi$. Prove that $E = \bigcup_{\alpha \in A} E_\alpha$ is connected.

4. (8 points) Let E be a non-empty compact subset of R^n. Prove that if f is continuous on E, then f is uniformly continuous on E.

5. (8 points) Given non-zero numbers $x_0, y_0, u_0, v_0, s_0, t_0$ that satisfy the simultaneous equations:

$$\begin{aligned} &u^2 + sx + ty = 0, \\ &v^2 + tx + sy = 0, \\ &2s^2x + 2t^2y - 1 = 0, \\ &s^2x - t^2y = 0, \end{aligned}$$

please prove that there exist functions $u(x, y)$, $v(x, y)$, $s(x, y)$, $t(x, y)$ and an open ball B containing (x_0, y_0) such that u, v, s, t are continuously differentiable and satisfy the above system on B such that $u(x_0, y_0) = u_0$, $v(x_0, y_0) = v_0$, $s(x_0, y_0) = s_0$, $t(x_0, y_0) = t_0$.

6. (8 points) Please prove that

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

(Hint: $\dfrac{1}{x} = \displaystyle\int_0^\infty e^{-xt} dt, \quad x > 0.$)

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(1) [10pts] Let A be a 3×3 complex matrix satisfying $A^3 = A^2$. Classify all such A's up to similarity.

(2) [10pts] Let V be a finite-dimensional vector space over $\mathbb{C}$. Let $A : V \to V$ be a linear transformation. Prove that A is diagonalizable if and only if the minimal polynomial of A has no multiple root.

(3) Let g be a 2×2 invertible complex matrix. Let $V = M_{2\times 2}(\mathbb{C})$ be the vector space of 2×2 complex matrices. Let $f : V \to V$ be the linear transformation $f(A) = gAg^{-1}$.

(a) [5pts] Suppose the eigenvalues of g are a and b with $a \neq b$. Compute the eigenvalues of f.

(b) [5pts] Suppose

$$g = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}.$$

Compute the eigenvalues of f.

(4) Let A, B be complex 2×2 matrices such that

$$\det(A) = \det(B) = 1.$$

(a) [5pts] Prove

$$\operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}) = 2\operatorname{tr}(A)\operatorname{tr}(B).$$

(b) [5pts] Prove

$$\operatorname{tr}(ABA^{-1}B^{-1}) = \operatorname{tr}(A)^2 + \operatorname{tr}(B)^2 + \operatorname{tr}(AB)^2 - \operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(AB) - 2.$$

(5) Let $P = \{f \in \mathbb{R}[x] : \deg(f) \leq n\}$, i.e P is the vector space of real polynomials of degrees less than or equal to n.

(a) [5pts] Let

$$D : P \longrightarrow P, \quad D(f) = \frac{df}{dx}.$$

Prove that D is a linear transformation on P. Compute the eigenvalues and the Jordan canonical form of D.

(b) [5pts] Prove that

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

is an inner product on P. For $n = 2$, find an orthonormal basis and compute D^*, the adjoint of D as defined in part (a).

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1. Let $A_n \subset \mathbb{R}^2$ be the interior of the circle with radius 2 and center at $(0, 1 + \frac{(-1)^n}{n})$. Find $\limsup_n A_n$ and $\liminf_n A_n$. (10 points)
(Recall that: $\limsup_n A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ and $\liminf_n A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$.)

2. Let μ be a finite measure on the σ-field $\mathcal{F}$. If $A_n \in \mathcal{F}, n = 1, 2 \cdots$, and $A_n \uparrow A$, show that $\mu(A) = \lim_{n\to\infty} \mu(A_n)$. (10 points)

3. If $\mu(\Omega) < \infty$, show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$. Give an example to show that this fails if $\mu(\Omega) = \infty$. (10 points)

4. Show that $\int_1^\infty e^{-t} \ln t \, dt = \lim_{n\to\infty} \int_1^n [1 - (t/n)]^n \ln t \, dt$. (10 points)

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Answer all the problems.

1. (10 pts) Let X be a finite set and H a finite p-group. Suppose H acts on X. Show that
$$|X| \equiv |X_0| \mod p,$$
where $X_0 = \{x \in X \,|\, gx = x \text{ for all } g \in G\}$.

2. (10 pts) Let G be a group of order 12 such that G has no elements of order 6. Show that the Sylow 2-subgroup of G is normal in G.

3. (10 %) An ideal M of a ring R is called a maximal ideal if for any ideal J such that $M \subset J \subset R$, then $J = M$ or $J = R$.

   Let R be a commutative ring with identity. Show R/M is a field if and only if M is maximal.

4. (10 %) Let E and F be fields.

   (a) Show that any finite extension E over F is also an algebraic extension over F.

   (b) Let α and β be algebraic over F. Show that for any $a, b \in F$, $a\alpha + b\beta$ is also algebraic over F.

   Note: E is a finite extension over F if $\dim_F E < \infty$ and E is algebraic over F if for any $a \in E$, there exists a non-zero polynomial $f(x)$ over F such that $f(a) = 0$.

— **The End** —