一、本試題分「分析通論」、「基礎代數」各佔 50 分，共 100 分。
二、作答時請務必在所屬答案卷上作答並標明題號。

分析通論

1. Assume that $\{x_\gamma : \gamma \in \mathbb{N}\}$ is a sequence of numbers of $\mathbb{R}$. A series $\sum_{\gamma=1}^{\infty} y_\gamma$ is called a *rearrangement* of $\sum_{\gamma=1}^{\infty} x_\gamma$ if and only if there exists a one-to-one surjective map $\mathbb{N} \xrightarrow{\phi} \mathbb{N}$ such that $y_\gamma = x_{\phi(\gamma)}$ $\forall \gamma \in \mathbb{N}$.

6% (a) Prove that the series $\sum_{\gamma=1}^{\infty} \frac{(-1)^{\gamma+1}}{\gamma}$ converges to $\ln 2$.

8% (b) Prove that there exists a rearrangement $\sum_{\gamma=1}^{\infty} y_\gamma$ of $\sum_{\gamma=1}^{\infty} \frac{(-1)^{\gamma+1}}{\gamma}$ such that $\sum_{\gamma=1}^{\infty} y_\gamma$ converges to 2008. [Note that the (alternating) series $\sum_{\gamma=1}^{\infty} \frac{(-1)^{\gamma+1}}{\gamma}$ converges, though $\sum_{\gamma=1}^{\infty} \frac{1}{\gamma}$ diverges.]

8% 2. Assume that $\{x_\gamma \in \mathbb{R} : \gamma \in \mathbb{N}\}$ is a *bounded* sequence of numbers of $\mathbb{R}$. Suppose that $\{y_\gamma \in \mathbb{R} : \gamma \in \mathbb{N}\}$ is a sequence *bounded from below* so that $\liminf_{\gamma \in \mathbb{N}} y_\gamma$ exists in $\mathbb{R}$. Prove that

$$\liminf_{\gamma \in \mathbb{N}} x_\gamma + \liminf_{\gamma \in \mathbb{N}} y_\gamma \le \liminf_{\gamma \in \mathbb{N}} \left(x_\gamma + y_\gamma\right) \le \limsup_{\gamma \in \mathbb{N}} x_\gamma + \liminf_{\gamma \in \mathbb{N}} y_\gamma.$$

3. Given a Lebesgue measurable subset A of $\mathbb{R}^n$, we denote by $|A|$ the Lebesgue measure of A. Assume that $\{E_\gamma : \gamma \in \mathbb{N}\}$ is a *decreasing* sequence of measurable subsets of $\mathbb{R}^n$ so that $E_\mu \supset E_\nu$ whenever $\mu \le \nu$. We define $\mathcal{E} = \bigcap_{\gamma \in \mathbb{N}} E_\gamma$.

3% (a) Suppose that $|E_\gamma|$ *is finite* for some $\gamma \in \mathbb{N}$. Prove that $|\mathcal{E}| = \lim_{\gamma \to +\infty} |E_\gamma|$.

3% (b) Show that the equality $|\mathcal{E}| = \lim_{\gamma \to +\infty} |E_\gamma|$ could be wrong when $|E_\gamma| = \infty$ $\forall \gamma \in \mathbb{N}$.

4. Given a Lebesgue measurable subset A of $\mathbb{R}^n$ we denote by $|A|$ the Lebesgue measure of A. Given a Lebesgue integrable function g on $\mathbb{R}^n$ the Hardy-Littlewood maximal function M_g of g is defined on $\mathbb{R}^n$ by

$$M_g(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g|$$

where $B_r(x)$ is the open ball with radius r centered at $x \in \mathbb{R}^n$.

6% (a) Prove that for each $t \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : M_g(x) > t\}$ is open.

8% (b) It follows from (a) that M_g is a measurable function on $\mathbb{R}^n$. Prove that when $\int_{\mathbb{R}^n} |g| > 0$ we have $\int_{\mathbb{R}^n} |M_g| = +\infty$.

8% 5. We say that a closed subset C of a metric space X is *nowhere-dense* if and only if C contains *no* nonempty open subset of X. Prove the Baire Category Theorem: When X is a complete metric space, there does *not* exist a countable collection $\{C_\gamma \subset X : \gamma \in \mathbb{N}\}$ of nowhere-dense closed subsets of X satisfying $X = \bigcup_{\gamma \in \mathbb{N}} C_\gamma$.

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基礎代數

Answer all the problems and show all your works.

1. (10%) Let G be a group of order 12 such that G has no elements of order 6. Show that G is isomorphic to A_4, the alternating group of degree 4.

2. (10%) Let R be a commutative ring with identity. Let A be an ideal of R such that A is contained in a finite union of prime ideals $P_1 \cup P_2 \cup \cdots \cup P_n$. Show that $A \subset P_j$ for some $j = 1, 2, \ldots, n$.

3. (10%) Let E be an extension field of K and L and M intermediate fields. Suppose that L is a finite Galois extension of K. Show that LM is a finite Galois extension of M. Moreover, $Gal(L/K) \cong Gal(LM/M)$, where $Gal(X/Y)$ denotes the Galois group of X over Y.

4. (10%) Let E and F be fields. Suppose that $E = F(x)$, the field generated by x over F, where x is transcendental over F

   (a) Let $F \subset K \subset E$ be an intermediate field such that $K \neq F$. Show that x is algebraic over K.

   (b) Let $y = \frac{f(x)}{g(x)} \in E$ with relatively prime $f(x), g(x) \in F[x]$. Find the degree $[F(x) : F(y)]$.

5. (10 %) Let V be a complex finite dimensional vector space of dimension n. Let ϕ and ψ be endomorphisms of V such that $\phi \circ \psi = \psi \circ \phi$. Show that there exists vector subspaces

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

such that $\dim V_i = i$, $\phi(V_i) \subset V_i$, and $\psi(V_i) \subset V_i$.

THE END