Part I.

NOTE: Be sure to show all work and explain your reasoning as clearly as possible.

1. (10 points) Let f, g be two Lebesgue integrable functions on $[0,1]$. Let $F(x) = \int_x^1 f(t)dt$, and $G(x) = \int_x^1 g(t)dt$. Prove that $\int_0^1 F(x)g(x)dx = F(0)G(0) - \int_0^1 G(x)f(x)dx$.

2. (10 points) Let $f_n \to f$ in $L^p(R^m)$, $1 \le p \le \infty$. And let $< g_n >$ be a sequence of measurable functions such that $|g_n| \le M$ for all n and $g_n \to g$ a.e. . Prove that $g_n f_n \to gf$ in $L^p(R^m)$.

3. (10 points) Assume that $f_n \in C^1([0,1])$ is a sequence such that $\|f_n\|_\infty \le 2^{-n}$ and $\|f_n'\|_\infty \le 2^{\frac{n}{2}}$. Let $f = \sum f_n$. Prove that there exists a constant $K < \infty$ such that

$$|f(x) - f(y)| \le K|x-y|^{\frac{1}{2}}, \text{ for all } x, y \in [0,1].$$

4. (10 points) Let A and B be Lebesgue measurable subsets of R^1 with $|A| > 0$ and $|B| > 0$. Prove that the set

$$\{x : x = x_1 - x_2, x_1 \in A, x_2 \in B\}$$

contains a nontrivial interval.

5. (10 points) Let $F(y) = \int_0^\infty e^{-x^2} \cos 2xy \, dx$, $y \in R$. Please calculate $F'(y)$ and deduce that

$$F(y) = \frac{1}{2}\sqrt{\pi}\, e^{-y^2}.$$

Part II.

6. (a) (5%) Suppose all elements of a finite group G is of order 2. Show that G is abelian.

   (b) (5%) Let G be a finite group. Let $a, b \in G$ be elements of order 2. Suppose that ab is of order 3. Show that the subgroup $H = < a, b >$ generated by a and b is isomorphic to the symmetry group S_3.

7. An integral domain D with an identity e is called a Euclidean domain if there is a function $d : D \setminus \{0\} \to \mathbb{Z}^+$ such that
   (I) $d(a) \le d(ab)$ for any $a, b \in D \setminus \{0\}$ and
   (II) for any $a \in D$ and $b \neq 0$, there are $q, r \in D$ such that $a = qb + r$, where $d(r) < d(b)$ or $r = 0$.

   (a) (5%) Show that $d(a) = d(e)$ if and only if a is an invertible element in D.

   (b) (5%) Show that every ideal of D is generated by one element.

8. Let E be a field extension over F. An element $v \in E$ is algebraic over F if there exists a polynomial $f(x)$ over F such that $f(v) = 0$. E is algebraic over F if for any element $v \in E$, v is algebraic over F.

   (a) (5%) Show that a finite extension E of F is also an algebraic extension of F.

   (b) (5%) Suppose that E is algebraic over K and K is algebraic over F. Show that E is algebraic over F.

9. Let V be a complex finite dimensional vector space and let ϕ and ψ be endomorphisms of V such that $\phi \circ \psi = \psi \circ \phi$.

   (a) (5%) Let λ be an eigenvalue of ϕ and let $E_\lambda = \{v \in V \mid \phi v = \lambda v\}$ be the eigenspace of ϕ of eigenvalue λ. Show that $\psi(E_\lambda) \subset E_\lambda$.

   (b) (5%) Show that there is a basis B of V such that the matrices of ϕ and ψ with respect to B are both upper triangular.

10. Let G be a group of order 56 with no element of order 14.
    (a) (5%) Prove that G has no normal subgroup of order 7.

    (b) (5%) Prove that G has a normal subgroup of order 8.