1. Let A be

$$A \equiv \left\{ (x,y) : x, y \in [0,1], x = \sum_{n=1}^{\infty} \frac{a_n}{7^n}, a_n = 2,5 \quad y = \sum_{n=1}^{\infty} \frac{b_n}{5^n}, b_n = 1,3 \right\}$$

Prove or disprove: i) A is closed; ii) A is open; iii) A is countable; iv) A is dense in $[0,1]$; v) A is Borel measurable. If A is Lebesgue measurable, find the measure $\mu(A)$. (10%)

2. Let f be in $L^1(I,\mu)$ and let S be $\{x : x \in I, f(x) \in Z\}$. Apply the bounded convergence theorem to show that

$$\lim_{n\to\infty} \int_I |\cos(\pi f(x))|^n dx = \mu(S)$$ (10%)

3. Let X be the space of continuous and 2π periodic functions. Show that

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)dx, \qquad f, g \in X$$

define an inner product in X. According to the mean square approximation we approximate $f(x) \in X$ by the trigonometric series

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

Choose proper coefficients a_k, b_k such that the integral (mean square error)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_n(x)|^2 dx$$

attains the minimum. (10%)

4. If f and g measurable in $\mathbb{R}^n$, their convolution is defined by

$$(f * g)(x) \equiv \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

provided the integral exists. Show that if $f \in L(\mathbb{R}^n)$ and $g \in L(\mathbb{R}^n)$ then $(f*g)(x)$ exists for almost every $x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |f * g| dx \le \left(\int_{\mathbb{R}^n} |f| dx\right)\left(\int_{\mathbb{R}^n} |g| dx\right)$$

Furthermore, if g is bounded and uniformly continuous on $\mathbb{R}^n$, then the convolution $f * g$ is bounded and uniformly continuous on $\mathbb{R}^n$. (10%)

5. Let $C[a,b]$ be the space of all continuous function on the closed interval $[a,b]$ and define

$$\|f\|_\infty \equiv \max_{x\in[a,b]} |f(x)|$$

Show that $\|\cdot\|_\infty$ is a norm in $C[a,b]$. Is $C[a,b]$ a Banach space? separable? Let $D = \{f \in C[a,b] \mid \|f\|_\infty \le 1\}$. Is D closed? bounded? compact? Can you describe all compact subsets of $C[a,b]$? Let $C^1[a,b]$ be the space of all continuous differentiable function on the closed interval $[a,b]$. Is $C^1[a,b]$ complete under this norm $\|\cdot\|_\infty$?

(10%)

6. Brief explanation for your examples is required. (10%)

   (i) Give an example of a group G and a normal subgroup H of G such that G does not have any subgroup isomorphic to G/H.

   (ii) Let G be the group of all invertible n by n matrices over $\mathbb{R}$. Find a nontrivial normal subgroup of G.

   (iii) Is it possible that there is an infinite field with finite characteristic? Why or why not?

   (iv) Give an example of a ring which is Noetherian but not Artinian.

7. (10%)

   (i) Let G be the finite group $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$. Find a one-to-one homomorphism from G into a symmetric group.

   (ii) Let R be the category of rings and ring homomorphisms. Construct a covariant functor $R \to R$ that assigns to each ring N the polynomial ring $N[x]$.

8. An integral domain R is called an *Euclidean domain* if there is a mapping τ from the set of nonzero elements of R to $\mathbb{N}$ that satisfies the following two conditions: (1) Given a and $b \neq 0$ in R, there exist q and r in R such that $a = bq + r$ and either $r = 0$ or $\tau(r) < \tau(b)$. (2) If $a \neq 0$ and $b \neq 0$ in R, then $\tau(ab) \geq \tau(a)$. Let R be an Euclidean domain. Call R ***residually finite*** if R/I is a finite ring for all nontrivial ideal I of R. (10%)

   (i) If $\{ r \in R \mid \tau(r) = k \}$ is a finite set for all $k \in \mathbb{N}$, show that R is residually finite.

   (ii) Show that $\mathbb{Z}(\sqrt{-1})$ is residually finite.

   (iii) Give an example of an Euclidean domain which is not residually finite.

9. (10%)

   (i) Explain why an angle of $90°$ can be trisected by ruler and compass constructions.

   (ii) Explain why an angle of $60°$ can not be trisected by ruler and compass constructions.

10. Let $\mathbb{C}^{\times}$ denote the multiplicative group of all nonzero complex numbers. Let G be a finite group. A homomorphism $\chi : G \to \mathbb{C}^{\times}$ is called a *character* of G. A character χ is called *trivial* if $\chi(g) = 1$ for all $g \in G$. (10%)

    (i) Suppose that G is the cyclic group of order 5. Give an example of a nontrivial character χ of G.

    (ii) Let G and χ be as in (i). Show that $\sum_{g \in G} \chi(g) = 0$.

    (iii) Let G be any finite group, and let χ be any character of G. Show that $\sum_{g \in G} \chi(g)$ is zero if and only if χ is not trivial.