PhD Entrance Exam, Elementary Math, June 12, 2001

Show all works

1. State the definition of the Lebesgue measure. Begin by defining outer measure and measurable sets. (10%)

2. Given a set $E \subset [0,1]$ with positive Lebegue measure $m(E) > 0$ and define $f(x) = m(E \cap [0,x])$ for $x \in [0,1]$. Prove that f is differentiable a.e. and compute f' (a.e.) on E. (10%)

3. Let $f \in L^1[0,1]$. Prove that $\forall \epsilon > 0, \exists \delta > 0$ such that $\int_E |f(x)|dx < \epsilon$ for every measurable set E for which $m(E) < \delta$. (10%)

4. Given f, $f_n \in L^1[0,1]$. $n = 1,2,3,\cdots$. Assume that $\sup_n \|f_n\|_1 < \infty$ and $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in [0,1]$. Prove or give a counterexample for the followings. $\lim_{n\to\infty} \int_0^1 (f_n(x) - f(x))g(x)\,dx = 0$ $\forall g \in L^\infty[0,1]$. (10%)

5. Let E be a Lebesgue measurable set on R^1 with a finite Lebesgue measure $m(E) < \infty$. For each $x \geq 0$, we define $f(x) = m(E \cap E_x)$, where $E_x = \{x + y | \, y \in E\}$. Prove that (10%)

(i) f is continuous on $[0, \infty)$.

(ii) $\lim_{x\to\infty} f(x) = 0$.

6. Brief explanations for your examples are required. (10%)

(i) Give an example of a non-abelian solvable group.

(ii) Given an example of an ideal I of a commutative ring R such that I is prime but not maximal.

(iii) Give an example of a unique factorization domain but not a principal ideal domain.

(iv) Give an example of a group algebra FG where F is a field and G is a group such that FG is not a division algebra.

7. Let G be a group. A group homomorphism $\pi: G \to \mathrm{GL}_n(\mathbf{C})$ is called an n-dimensional representation of G where $\mathrm{GL}_n(\mathbf{C})$ is the group of all n by n matrices over $\mathbf{C}$ with non-zero determinants. (10%)

(i) Show that the commutator subgroup of G is contained in the kernel of π when $n = 1$.

(ii) Given an example of a two-dimensional representation of the group $\mathbf{Z}/7\mathbf{Z}$.

(iii) Given an example of a two-dimensional representation of the symmetric group S_3.

8. (10%)

(i) Show that a field is an Euclidean domain.

(ii) Let R be a commutative ring (with unity). An element $a \in R$ is called a nilpotent element if $a^n = 0$ for some $n \in \mathbf{N}$. Show that the set of all nilpotent elements in R forms an ideal.

9. Let $M_2(\mathbf{R})$ denote the algebra of two by two matrices over $\mathbf{R}$. Let $\mathbf{H}$ denote the quaternion algebra i.e., $\mathbf{H}$ is the algebra over $\mathbf{R}$ with basis $\{1, i, j, k\}$ and the relation $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. (10%)

(i) Show that $M_2(\mathbf{R})$ and $\mathbf{H}$ are not isomorphic as algebras over $\mathbf{R}$.

(ii) Show that $M_2(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{C}$ are isomorphic as algebras over $\mathbf{C}$.

10. Let ϕ be a field automorphism of real number field $\mathbf{R}$. (10%)

(i) Show that $\phi(a) = a$ for any rational number a.

(ii) Show that $\phi(a) > \phi(b)$ if $a, b \in \mathbf{R}$ and $a > b$.

(iii) Show that ϕ is the identity map.