(1) Let V be a real vector space.

10% (a) Suppose that $J: V \to V$ a linear map such that $J^2 = -I$, where I is the identity map. Prove that $\dim V$ is an even integer.

10% (b) Let J_0 be the $2n \times 2n$ matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, where I_n is the $n \times n$ identity matrix. Let A be a $2n \times 2n$ matrix such that $A^t J_0 A = J_0$. Prove that λ is an eigenvalue of A if and only if $\frac{1}{\lambda}$ is an eigenvalue of A.

(2) Let A be any $n \times n$ matrix. We let e^A be the matrix $\sum_{i=0}^{\infty} \frac{A^i}{i!}$.

5% (a) Prove that $AB = BA$ implies that $e^{A+B} = e^A e^B$.

5% (b) Prove that e^A is invertible for every A. What is $(e^A)^{-1}$?

5% (c) Prove that $\det(e^A) = e^{\mathrm{Tr}A}$.

(3) Let A be a complex $n \times n$ matrix. Prove that there exist complex matrices A_s and A_n with the following properties:

7% (a) A_s is diagonalizable and A_n is nilpotent.

8% (b) $A_s A_n = A_n A_s$ and $A = A_s + A_n$.

8% (4) Let $G = \langle x, y \mid xy = y^2 x, yx = x^2 y \rangle$. Find $|G|$.

(5) Let p be a prime number.

12% (a) Let P be a p-group. Let A be a normal subgroup of order p. Prove that A is contained in the center of P.

12% (b) Let G be a finite group and H a normal subgroup of G. Assume that the order of H is p. Prove that H is contained in every p-Sylow subgroup of G.

18% (6) Let A be a finite dimensional algebra over a field F. If A contains no nonzero zero-divisors, then A is a skew-field.

(7) Let $\{f_n\}$ be a sequence of absolutely continuous functions which converges to a function f uniformly. Is f absolutely continuous? Prove it or give a counterexample! (10pt)

(8) Assume that $X = \{a, b\}$ and μ is a measure defined on X by $\mu(a) = 1$, $\mu(b) = \mu(X) = \infty$, and $\mu(\phi) = 0$. Is $L^\infty(\mu)$ the dual space of $L^1(\mu)$? State your reasons. (20pt)

(9) Prove that for $a_{ij} \geq 0$, $\left(\sum_i (\sum_j a_{ij})^p \right)^{\frac{1}{p}} \leq \sum_j \left(\sum_i a_{ij}^p \right)^{\frac{1}{p}}$. (20pt)

(10)(a) Show that in a metric space $< X, \rho >$, the function $f(x) = \rho(x, x_0)$ is a continuous function. (5pt)

(b) Show that in the metric space $< R^2, \rho >$, where $\rho(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, the set $S = \{(x, y) : \sqrt{(x-1)^2 + (y-a)^2} < 1\}$ is open. (5pt)

$= \{(x,y) : |x-1| + |y-a| < 1\}$

(11)(a) Show that $\lim_{p \to \infty} \| < x_n > \|_{l^p} = \| < x_n > \|_{l^\infty}$. If you think this is wrong, please give a counterexample. (20pt)

(b) Let $f \in L^p(\mu)$. Show that the set $N = \{x : f(x) \neq 0\}$ is σ-finite. (20pt)