國立成功大學 103 學年度「博士班」研究生甄試入學考試

【高等微積分】

advanced calculus

1. Define a function f by

$$f(x) = \begin{cases} x \cdot \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove or disprove that f is uniformly continuous on $\mathbb{R}$. (20 points)

2. Let f be a smooth function on $(-1, 1)$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-1, 1)$. Let $\{x_m\}$ be a sequence with $x_m \neq 0$ for all $m \in \mathbb{N}$. Assume that $\{x_m\}$ converges to zero with $f(x_m) = 0$. Show that $f = 0$ on $(-1, 1)$. (20 points)

3. Let f be continuous on $[0, 1]$ and suppose that $f(0) = 0$. Prove that

$$\lim_{n \to \infty} \int_0^1 f(x^n) = 0.$$

(20 points)

4. Evaluate $\int_{\mathbb{R}^n} \|x\|^4 \cdot e^{-\|x\|^2} dx$. (20 points)

5. Consider the inhomogeneous initial value problem

$$\begin{cases} y''(t) + \omega^2 y(t) = q(t), \\ y(0) = y'(0) = 0. \end{cases} \tag{0.1}$$

For each $s \leq t$, let $z(t, s)$ be the solution of the homogeneous initial value problem

$$\begin{cases} z_{tt} + \omega^2 z = 0, \\ z(s, s) = 0, \quad z_t(s, s) = q(s). \end{cases} \tag{0.2}$$

Show that

$$u(t) := \int_0^t z(t, s) ds$$

is the solution of (0.1). (20 points)

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【線性代數】

Notation

- n: a positive integer
- $\mathrm{M}_{n\times n}(F)$: the set of all $n \times n$ matrices over the field F
- $\mathbb{R}$: the field of all real numbers
- $\mathbb{C}$: the field of all complex numbers
- A^*: the conjugate transpose of the matrix A

1. (10%) Let V be a 2013-dimensional real vector space, and let W be a 13-dimensional subspace of V. Show that there exist linear functionals $T_1, \ldots, T_{2000}$ on V such that $W = \mathrm{N}(T_1)\cap\cdots\cap\mathrm{N}(T_{2000})$. (Here $\mathrm{N}(T_i)$ denotes the null space of T_i for $i = 1, \ldots, 2000$.)

2. (12%) Show that if A is a 3×3 real matrix, then A is similar to
$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & \mu \\ 0 & 1 & \nu \end{pmatrix}$$
for some $\lambda, \mu, \nu \in \mathbb{R}$.

3. (12%) Let A be a 6×6 complex matrix such that $A^3 = 0$. Find all possible Jordan canonical forms of A.

4. (10%) Let G be the multiplicative group of all invertible $n \times n$ complex matrices. Show that every element of finite order in G is diagonalizable.

5. (12%) Suppose T is a diagonalizable linear operator on a finite-dimensional real vector space V and W is a T-invariant subspace of V. Let $\lambda_1, \ldots, \lambda_k$ be all distinct eigenvalues of T, and let $W_{\lambda_i} = \{w \in W \mid T(w) = \lambda_i w\}$ for all i. Show that $W = \bigoplus_{i=1}^{k} W_{\lambda_i}$.

6. (12%) Suppose $N \in \mathrm{M}_{n\times n}(\mathbb{C})$ is normal, i.e., $N^*N = NN^*$. Show that N is self-adjoint if and only if all eigenvalues of N are real.

7. Let $\langle A, B\rangle$ be the trace of AB^* for all $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$.

   (a) (10%) Show that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathrm{M}_{n\times n}(\mathbb{C})$.

   (b) (10%) Let $P \in \mathrm{M}_{n\times n}(\mathbb{C})$ be invertible, and let T be the linear operator on $\mathrm{M}_{n\times n}(\mathbb{C})$ defined by $T(A) = P^{-1}AP$. Find the adjoint of T with respect to the inner product $\langle\cdot,\cdot\rangle$.

8. (12%) Let $U(n)$ be the set of all $n \times n$ unitary (complex) matrices, and let $\mathfrak{u}(n)$ be the set of all $n \times n$ complex matrices A with $A^* = -A$. There is a well-defined function $e\colon \mathfrak{u}(n) \to U(n)$ given by
$$e(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$
Show that e is surjective.