成功大學 100 學年度博士班研究生甄試入學考試

【高等微積分】

1. Suppose that $x \in R$, $x_n \geq 0$, and $x_n \to x$ as $n \to \infty$. Prove that $\sqrt{x_n} \to \sqrt{x}$ as $n \to \infty$.

2. Suppose that I is a closed, bounded interval of R and $f : I \to R$ is continuous on I. Please prove that f is uniformly continuous on I.

3. Let
$$u(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}, \quad t > 0, x \in R.$$
   (a) Prove that u satisfies the heat equation: $u_{xx} - u_t = 0$ for all $t > 0$ and $x \in R$.
   (b) If $a > 0$, prove that $u(x,t) \to 0$ as $t \to 0+$, uniformly for $x \in [a, \infty)$.

4. Please compute the following integrals.
   (a) $\int_0^1 \int_0^1 \sqrt{xy + x}\, dxdy$.
   (b) $\int_0^{\pi/2} e^x \sin x dx$.

5. Determine whether the following series converges or diverges.
   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{\log k}{k^p}, \quad p > 1.$
   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{9k^2 + 3}{k^3 - 2k + 1}.$

6. Let A and B be subsets of R^n. Prove that
$$\partial(A \cap B) \subseteq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B).$$

7. Evaluate the following expression:
$$\frac{d}{dy} \int_{-1}^{1} \sqrt{x^2y^2 + xy + y + 2}\, dx \quad \text{at } y = 0.$$

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Linear Algebra PhD Entrance Exam Date: 28/10/2010

Work out all problems and no credit will be given for an answer without reasoning.

1. (a) (5%) Let V be the vector space of n-square matrices over K. Let M be an arbitrary matrix in V. Let $T: V \to V$ be defined by $T(A) = AM + MA$, where $A \in V$. Show that T is a linear transformation.

   (b) (5%) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by

   $$T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$$

   Find a basis and the dimension of the kernel W of T. What is the dimension of the image U of T?

   (c) (5%) Show that no matrices A and $B \in M_{n \times n}(F)$ such that $AB - BA = I$, where I is an $n \times n$ identity matrix.

2. (a) (5%) Show that if A is a self-adjoint matrix, then all eigenvalues of A are real.

   (b) (10%) Let V be the vector space of n-square matrices over a field $\mathbb{R}$. Let U and W be the subspaces of symmetric and skew-symmetric matrices, respectively. Show that $V = U \oplus W$.

3. (a) (7%) Let

   $$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

   Find A^n.

   (b) (8%) Find $\det(A^{-1})$ for

   $$A = \begin{bmatrix} 1+x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & 1+x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & 1+x_3 & \dots & x_n \\ \dots & \dots & \dots & \dots & \dots \\ x_1 & x_2 & x_3 & \dots & 1+x_n \end{bmatrix}.$$

4. Let

   $$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

   (a) (5%) Find the characteristic polynomial of A.

   (b) (5%) Find the minimal polynomial of A.

   (c) (5%) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

5. (a) (7%) Let T be a linear operator on a finite dimensional inner product space V. Show that there exists a unique linear operator T^* on V such that $\langle T(u), v\rangle = \langle u, T^*(v)\rangle$, for every $u, v \in V$.

   (b) (8%) Let V be a finite-dimensional inner product space, and let E be an idempotent linear operator on V, i.e., $E^2 = E$. Prove that E is self-adjoint if and only if $EE^* = E^*E$.

6. (10%) Let V be a finite dimensional vector space over a field F. Let V^* and V^{**} be the dual and double dual space of V. Define $\Phi : V \to V^{**}$ by

$$\Phi(x)(f) = f(x) \quad \text{for all } x \in V,\ f \in V^*.$$

Show that Φ is an isomorphism of V onto V^{**}.

7. Let G be a group. A subgroup H of G is called a *normal subgroup* of G if $aH = Ha$ for all a in G. Let G' be the subgroup of G generated by the set $S = \{x^{-1}y^{-1}xy | x, y \in G\}$.

   (a) (5%) Prove that G' is normal in G.

   (b) (5%) Prove that G/G' is Abelian.

   (c) (5%) Prove that if H is a subgroup of G and $G' \leq H$, then H is normal in G and G/H is Abelian.