國立成功大學一〇〇學年度 博士班入學考試 高等微積分 試題 共 1 頁

注意事項：作答時請務必在所屬答案卷上作答並標明題號。	100.05.03

1. Let the sequence $\{a_n\}$ be given recursively by the formula

$$\begin{cases} a_1 = 2, \\ a_{n+1} = (2a_n + 4)/3. \end{cases}$$

Show that $\{a_n\}$ is monotone increase and bounded above. (15 points)

2. Show that the function

$$f(x) = \int_0^{1/x} \frac{1}{1+t^2}dt - \int_x^0 \frac{1}{1+t^2}dt$$

is constant for $x > 0$. (15 points)

3. Prove that

$$|\int_0^1 x \cdot \sin(1/x)\, dx| \le (\int_0^1 x^2 \cdot \sin^2(1/x)\, dx)^{1/2}.$$

(Hint:$(\sum a_k b_k)^2 \le \sum a_k^2 \sum b_k^2$.) (15 points)

4. Let f be continuous on the interval $[0, b]$ where $f(x) + f(b-x) \neq 0$ on $[0, b]$. Show that

$$\int_0^b \frac{f(x)}{f(x)+f(b-x)}dx = \frac{b}{2}.$$

(15 points)

5. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies

$$f^2(t) = 2\int_0^t f(s)ds$$

for $t > 0$. Show that either $f \equiv 0$ or there is a $t_0 \ge 0$ such that

$$f(t) = \begin{cases} t - t_0, & t \ge t_0, \\ 0, & 0 \le t \le t_0. \end{cases}$$

(20 points)

6. Let $f(x)$ be bounded and continuous on $[0, \infty)$. Let

$$F(t) = \int_0^\infty \frac{tf(x)}{t^2 + x^2}dx.$$

Find $\lim_{t\to 0^+} F(t)$. (20 points)

國立成功大學一〇〇學年度 博士班入學考試 線性代數 試題 共1頁

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Time 9:50–11:00

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, real numbers and complex numbers, repectively.

1. (10%) Let A be an $n \times m$ matrix with entries in $\mathbb{R}$ and let I_m denote the $m \times m$ identity matrix. Prove that there is an $m \times n$ matrix B such that $BA = I_m$ if and only if the rank of the matrix A is m.

2. (15%) Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ and $T : V \to W$ be a linear transformation. Let V^* and W^* denote the dual spaces of the vector spaces V and W, and let $T^* : W^* \longrightarrow V^*$ be a linear transformation defined by $T^*(f) = f \circ T$. Prove that $T : V \to W$ is onto if and only if $T^* : W^* \to V^*$ is one-to-one.

3. (30%) Let W be a subspace of the set of functions from $\mathbb{R}$ to $\mathbb{R}$ spanned by the set $\mathscr{B} = \{1, e^x, xe^x, x^2e^x\}$. Let $T : W \longrightarrow W$ be a linear transformation defined by
$$T(f)(x) = \frac{d}{dx}f(x) - f(x).$$
   (a) (10%) Show that $\mathscr{B} = \{1, e^x, xe^x, x^2e^x\}$ is a linear independent set.
   (b) (5%) Find the matrix representation of T with respect to the basis $\mathscr{B}$.
   (c) (5%) Find the matrix representation of T^2 with respect to the basis $\mathscr{B}$.
   (d) (5%) Find a basis for the kernel of T^2.
   (e) (5%) Find a basis for the image of T^2.

4. (25%)
   (a) (15%) Let V be a finite dimensional vector space over $\mathbb{R}$ with an inner product $\langle\, ,\rangle$. Let $\{w_1, w_2, \cdots, w_k\}$ be an orthonormal basis for a subspace W of V with $k \in \mathbb{N}$. For $v \in V$, we let $\|v\| = \sqrt{\langle v, v\rangle}$. Fix a vector $u \in V$ and define $d = \inf\{\|u - w\| \in \mathbb{R} \mid w \in W\}$. Prove that $d = \|u - \langle u, w_1\rangle w_1 - \langle u, w_2\rangle w_2 \cdots - \langle u, w_k\rangle w_k\|$.
   (b) (10%) Find $\alpha, \beta \in \mathbb{R}$ so that
$$\int_0^\pi (\cos x - \alpha x - \beta)^2\, dx \le \int_0^\pi (\cos x - ax - b)^2\, dx, \qquad \text{for all } a,\ b \in \mathbb{R}.$$

5. (20%) Let V be a finite dimensional vector space over $\mathbb{C}$ with the hermitian inner product $\langle\, ,\rangle$. Let $\mathscr{B} = \{v_1, v_2, \cdots, v_n\}$ be an orthonormal basis for V. Let $T : V \longrightarrow V$ be a linear transformation such that $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is also an orthonormal basis for V.
   (a) (5%) Show that $\langle T(v), T(w)\rangle = \langle v, w\rangle$ for all $v,\ w \in V$.
   (b) (5%) Show that if $\{w_1, w_2, \cdots, w_n\}$ is any orthonormal basis for the vector space V, then $\{T(w_1), T(w_2), \cdots, T(w_n)\}$ is also an orthonormal basis for V.
   (c) (10%) Show that $TT^* = T^*T = id_V$, where id_V is the identity map of V and T^* is the adjoint operator with respect to the hermitian inner product.

國立成功大學一ＯＯ學年度 博士班入學考試 實變數函數論 試題 共 1 頁

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1. (10 points) Let f be a simple function, taking its distinct values $a_1, \dots, a_N$ on disjoint sets $E_1, \dots, E_N$, respectively. Show that $f(x) = \sum_{i=1}^{N} a_i \varkappa_{E_i}(x)$ is measurable on $E = \cup_{i=1}^{N} E_i$ if and only if $E_1, \dots, E_N$ are measurable.

2. (10 points) Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ be a nonnegative measurable function such that $\int_E f < \infty$. Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable subset $E_1 \subset E$ with $|E_1| < \delta$ we have $\int_{E_1} f < \varepsilon$.

   Hint: For each $k = 1, 2, \dots$, and $x \in E$, define the function $f_k(x) = \begin{cases} f(x) & \text{if } f(x) < k, \\ k & \text{if } f(x) \geq k. \end{cases}$

   Note that the sequence $0 \leq f_k(x) \nearrow f(x)$ on E.

3. (10 points) Let $f(x, y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $(\partial f(x, y)/\partial x)$ is a bounded function of (x, y). Show that $(\partial f(x, y)/\partial x)$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x, y)dy = \int_0^1 \frac{\partial}{\partial x} f(x, y)dy.$$

   Hint: For each $n = 1, 2, \dots$, define $F_n(x, y) = \dfrac{f(x + \frac{1}{n}, y) - f(x, y)}{\frac{1}{n}}$, and determine $\lim_{n\to\infty} F_n(x, y)$.

4. (10 points) Let $\phi(x)$, $x \in \mathbb{R}^n$ be a bounded measurable function such that $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi = 1$. For $\epsilon > 0$, let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\epsilon\to 0}(f * \phi_\epsilon)(x) = f(x)$$

   at each Lebesgue point x of f, i.e. x is a point at which

$$\lim_{Q\searrow x} \frac{1}{|Q|}\int_Q |f(y) - f(x)|dy = 0 \text{ is valid.}$$

   Hint: Note that $\lim_{\epsilon\to 0}(f * \phi_\epsilon)(x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} f(x - y)\phi_\epsilon(y)\,dy$ and $f(x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} f(x)\phi_\epsilon(y)\,dy$

5. (10 points) Let f, $\{f_k\} \in L^p$. Show that if $\|f - f_k\|_p \to 0$, then $\|f_k\|_p \to \|f\|_p$. Conversely, if $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$, $1 \leq p < \infty$, show that $\|f - f_k\|_p \to 0$.
   Hint: For the converse, you may use the following inequality and apply Fatou's lemma.

$$2^p |f|^p + 2^p |f_k|^p - |f - f_k|^p \geq 0$$

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Work out all of the problems and show details of your works.

1. Let p be a prime number and n a positive integer.

[6%] (a) Prove that any two groups of order p are isomorphic.

[5%] (b) Show that if G is a group of order p^n, then G has an element of order p.

[5%] (c) Show that if G is a group of order p^2, then G is abelian.

[5%] 2. (a) Find the centralizers in S_7 of the permutation $\sigma = (123)(4567)$.

[5%] (b) Give a list of representatives for the conjugacy classes of elements of order 6 in S_7.

3. Let $f : R \to S$ be a homomorphism of rights with kernel K and image I.

[6%] (a) Show that if N is a subring of R, then $f^{-1}(f(N)) = K + N$.

[6%] (b) Show that if L is a subring of S, then $f(f^{-1}(L)) = I \cap L$.

4. Let F be a field extension of the rational numbers.

[6%] (a) Show that the set $\{a + b\sqrt{2} \mid a, b \in F\}$ is a field.

[6%] (b) Give necessary and sufficient conditions for the set $L = \{a + b\sqrt[3]{2} \mid a, b \in F\}$ to be a field.