國立成功大學 103 學年度 微積分競試

2015 年 4 月 25 日

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說明:

1. 本試題含封面共 9 頁，8 大題。
2. 考試時間 **100** 分鐘。
3. 請在每個試題所屬的頁面作答。如欲使用試題背面，請標示清楚。
4. 請清楚地寫出計算及證明的過程，沒有過程的答案將不予記分。

題號	配分	分數
1	10	
2	15	
3	10	
4	20	
5	15	
6	10	
7	10	
8	10	
總分	100	

1. 計算下列積分

(a) (5 points)

$$\int_1^2 \ln x \, dx =$$

(a) __________

Solution:

$$\int_1^2 \ln x \, dx = (x \ln x)\Big|_1^2 - \int_1^2 x \cdot \frac{1}{x} \, dx = 2\ln 2 - 1$$

(b) (5 points)

$$\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}} =$$

(b) __________

Solution: Substitute x by $\frac{\sin\theta}{2}$. We get

$$\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}} = \int_0^{\frac{\pi}{6}} \frac{\frac{1}{2}\cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}} = \frac{\pi}{12}$$

2. 令

$$F(\theta) = \frac{1}{\frac{1}{2}\sin\theta + \cos\theta}, \; 0 \le \theta \le \frac{\pi}{2}.$$

(a) (5 points) 求 $F(\theta)$ 的臨界點 (critical point) $\hat{\theta}$，即 $F'(\theta) = 0$ 的解。

(a) ______________

Solution: $\hat{\theta}$ is a solution of $F'(\theta) = \frac{-\left(\frac{1}{2}\cos\theta - \sin\theta\right)}{\frac{1}{2}\sin\theta + \cos\theta} = 0 \implies \hat{\theta} = \tan^{-1}\frac{1}{2}$

(b) (5 points) 求 $F(\theta)$ 在 $(0, \frac{\pi}{2})$ 的相對極值 (relative/local extreme values) 並說明原因。

(b) ______________

Solution: Around $\theta = \hat{\theta}$ $F(\theta)$ changes from negative to positive, so F attains a local min at $\theta = \hat{\theta}$. $F(\hat{\theta}) = \frac{2}{\sqrt{5}}$.

(c) (5 points) 求 $F(\theta)$ 在 $[0, \frac{\pi}{2}]$ 的絕對極值 (absolute extreme values)。

(c) ______________

Solution: absolute min $F(\hat{\theta}) = \frac{2}{\sqrt{5}}$; absolute max $F(\frac{\pi}{2}) = 2$.

3. 令 S 為 $y=\sin x$, $x=\frac{\pi}{2}$, 及 $y=0$ 所圍成的區域。

   (a) (5 points) 求 S 對 x 軸旋轉的旋轉體體積。

(a) ______

Solution:

$$\int_0^{\frac{\pi}{2}} \pi(\sin x)^2\,dx = \pi\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx = \pi\left[\frac{x}{2}-\frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$$

   (b) (5 points) 求 S 對 y 軸旋轉的旋轉體體積。

(b) ______

Solution: Solution 1: volume =

$$2\pi\int_0^{\frac{\pi}{2}} x\sin x\,dx = 2\pi\left[-x\cos x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos x\,dx\right] = 2\pi.$$

Solution 2: volume= $\pi\cdot(\frac{\pi}{2})^2 - \int_0^1 \pi(\sin^{-1}y)^2\,dy = 2\pi$, because

$$\begin{aligned}\int_0^1 (\sin^{-1}y)^2\,dy = \int_0^{\frac{\pi}{2}} x^2\,d(\sin x) = x^2\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x\sin x\,dx \\ = \frac{\pi^2}{4} + 2\int_0^{\frac{\pi}{2}} x\,d(\cos x) \\ = \frac{\pi^2}{4} + 2x\cos x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\cos x\,dx \\ = \frac{\pi^2}{4} - 2.\end{aligned}$$

4. 令

$$F(x) = \int_0^x 3^t\,dt.$$

(a) (5 points) 證明 $F(x)$ 為絕對遞增 (strictly increasing) 函數。

(a) ______

Solution: By the fundamental theorem of calculus, $F'(x) = 3^x > 0$ for all x. So $F(x)$ is strictly increasing by the first derivative test.

(b) (5 points) 求不定積分

$$\int 3^t\,dt =$$

(b) ______

Solution:

$$\int 3^t\,dt = \int e^{(\ln 3)t}\,dt = \frac{3^x}{\ln 3} + C.$$

(c) (5 points) 求 $F^{-1}(\frac{2}{\ln 3})$，即求 $F(x) = \frac{2}{\ln 3}$ 的解。

(c) ______

Solution: Solve for $\frac{2}{\ln 3} = F(x) = \frac{3^x}{\ln 3} - \frac{1}{\ln 3}$, we get $x = F^{-1}(\frac{2}{\ln 3}) = 1$.

(d) (5 points) 求 $(F^{-1})'(\frac{2}{\ln 3})$。

提示：利用連鎖律 $F'(F^{-1}(x)) \cdot (F^{-1})'(x) = 1$，並將 $x = \frac{2}{\ln 3}$ 代入此等式。

(d) ______

Solution: Plugging $x = \frac{2}{\ln 3}$ in the hinted formula, we see that

$$1 = F'(F^{-1}(\frac{2}{\ln 3})) \cdot (F^{-1})'(\frac{2}{\ln 3}) = F'(1) \cdot (F^{-1})'(\frac{2}{\ln 3}).$$

Since $F'(1) = 3$, we conclude that $(F^{-1})'(\frac{2}{\ln 3}) = \frac{1}{3}$.

5. 令

$$f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ -1 < x < 1.$$

(a) (5 points) 求 $f'(x)$。

(a) ______________

Solution: $f'(x) = \frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{2}{(1-x)^2} = \frac{1}{1-x^2}$

(b) (5 points) 求 $f'(x)$ 在 $x=0$ 的泰勒級數 (Taylor series)。

(b) ______________

Solution: $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

(c) (5 points) 求 $f(x)$ 在 $x=0$ 的泰勒級數 (Taylor series)。

(c) ______________

Solution: Integrating $f'(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$, we get $f(x) = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$. Since $f(0) = 0$, the constant $C = 0$. Therefore,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.$$

6. (10 points) 判斷下列級數是否收斂並說明理由。

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

提示：計算瑕積分 (improper integral)

$$\int_{2}^{\infty} \frac{dx}{x \ln x}$$

Solution: The series diverges, because

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} > \int_{2}^{\infty} \frac{dx}{x \ln x} = \lim_{t \to \infty} \int_{\ln 2}^{t} \frac{du}{u} = \lim_{t \to \infty} (\ln t - \ln(\ln 2)) = \infty.$$

7. (10 points) 證明對所有實數 $x > y > 0$，下列不等式恆成立。

$$\sqrt{1+x} - \sqrt{1+y} < \frac{1}{2}(x-y)$$

Solution: Proof 1: By the mean value theorem, there exists $c \in (y, x)$ such that

$$\frac{\sqrt{1+x} - \sqrt{1+y}}{x-y} = \frac{1}{2}(1+c)^{\frac{-1}{2}} < \frac{1}{2}.$$

Proof 2: Consider the function $f(x) = \sqrt{1+x} - \frac{1}{2}x$. Since

$$f'(x) = \frac{1}{2}(1+x)^{\frac{-1}{2}} - \frac{1}{2} < 0$$

for all $x > 0$, the statement follows from the first derivative test.

8. 令

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) (3 points) 判斷 $f(x)$ 是否在 $x = 0$ 連續並說明原因。

Solution: Yes, because $\lim_{x\to 0} e^{-\frac{1}{x^2}} = 0$.

(b) (7 points) 判斷 $f(x)$ 是否在 $x = 0$ 可微分並說明原因。
提示：利用 L'Hospital's rule。

Solution: By L'Hopital's rule,

$$f'(0) = \lim_{x\to 0} \frac{1}{x} e^{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{(x^{-1})'}{(e^{\frac{1}{x^2}})'} = \lim_{x\to 0} \frac{-x^{-2}}{-2x^{-3}(e^{\frac{1}{x^2}})} = 0.$$