National Cheng Kung University

Calculus Exam

April 25, 2015

Name:__________________ Department:__________________

Student ID number:____________________________________

Instructions:

1. There are 9 pages (including the cover page) in this exam.
2. You have **100 minutes** to work on the exam.
3. Write your answers above the answer line, if an answer line is provided.
4. The computation processes/proofs of each problem is required. An answer without any explanations will not be graded.

Problem	Points	Score
1	10	
2	15	
3	10	
4	20	
5	15	
6	10	
7	10	
8	10	
Total	100	

1. Evaluate the following integrals.

   (a) (5 points)

$$\int_1^2 \ln x \, dx =$$

(a) ____________

   (b) (5 points)

$$\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}} =$$

(b) ____________

2. Let

$$F(\theta) = \frac{1}{\frac{1}{2}\sin\theta + \cos\theta}, \; 0 \le \theta \le \frac{\pi}{2}.$$

(a) (5 points) Find the critical point $\hat{\theta}$ of $F(\theta)$, i.e. solve $F'(\theta) = 0$ for θ.

(a) ______________

(b) (5 points) Find the relative/local extreme values of $F(\theta)$ in $(0, \frac{\pi}{2})$ and justify your answer.

(b) ______________

(c) (5 points) Find the absolute extreme values of $F(\theta)$ in $[0, \frac{\pi}{2}]$.

(c) ______________

3. Let S be the region bounded by $y = \sin x$, $x = \frac{\pi}{2}$, and $y = 0$.

   (a) (5 points) Find the volume of the solid obtained by rotating S about the x-axis.

   (a) ______________

   (b) (5 points) Find the volume of the solid obtained by rotating S about the y-axis.

   (b) ______________

4. Let

$$F(x) = \int_0^x 3^t \, dt.$$

(a) (5 points) Prove that $F(x)$ is strictly increasing.

(b) (5 points) Find the indefinite integral

$$\int 3^t \, dt =$$

(b) ______________

(c) (5 points) Find $F^{-1}(\frac{2}{\ln 3})$, i.e. solve $F(x) = \frac{2}{\ln 3}$ for x.

(c) ______________

(d) (5 points) Find $(F^{-1})'(\frac{2}{\ln 3})$.

Hint: Use the chain rule $F'(F^{-1}(x)) \cdot (F^{-1})'(x) = 1$, and plug $x = \frac{2}{\ln 3}$ into this equation.

(d) ______________

5. Let

$$f(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \; -1 < x < 1.$$

(a) (5 points) Find $f'(x)$.

(a) ____________

(b) (5 points) Find the Taylor series of $f'(x)$ at $x = 0$.

(b) ____________

(c) (5 points) Find the Taylor series of $f(x)$ at $x = 0$.

(c) ____________

6. (10 points) Determine whether the following series is convergent and justify your answer.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Hint: Compute the following improper integral

$$\int_{2}^{\infty} \frac{dx}{x \ln x}$$

7. (10 points) Prove that for all $x > y > 0$,

$$\sqrt{1+x} - \sqrt{1+y} < \frac{1}{2}(x-y).$$

8. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) (3 points) Determine whether $f(x)$ is continuous at $x = 0$ and justify your answer.

(b) (7 points) Determine whether $f(x)$ is differentiable at $x = 0$ and justify your answer.
Hint: Use the L'Hospital's rule ∘