MIDTERM 3 FOR ALGEBRA

Date: 1999, December 27, 10:10–11:00AM

Each of Problems 1–5 is worth 14 points, each of problems 6-7 is worth 15 points.

1.

(i) Let G be a group of order 540. What is the order of a Sylow 3-subgroup of G?

(ii) Find the characteristic of the ring $\mathbf{Z}_3 \times \mathbf{Z}_4$.

2.

(i) Let φ denote the Euler phi-function. Compute $\varphi(24)$.

(ii) Use Fermat's theorem to find the remainder of 37^{48} when it divided by 7.

3. Show that a Sylow 5-subgroup of a group G of order 15 is normal.

4. What is the free group $F[\{a\}]$? Show that $(a : a^7)$ is a presentation of $\mathbf{Z}_7$.

5. Show that if U is the collection of all units in a ring $\langle R, +, \cdot \rangle$ with unity, then $\langle U, \cdot \rangle$ is a group.

6. An element a of a ring R is *idempotent* if $a^2 = a$. Show that a division ring contains exactly two idempotent elements.

7. Prove that if D is an integral domain, then the ring of polynomials $D[x]$ is also an integral domain.