FINAL EXAM OF ALGEBRA

Date: 2000, June 19, 15:10–17:00

An answer without reasoning will not be accepted.

1. [4%–4%–4%–5%–5%–5%]

(i) Give an example of a finite group which is not solvable.

(ii) Give an example of a field which is not perfect.

(iii) Construct a field of order 9.

(iv) Explain why $\mathbf{Z}[x]$ is a UFD but not a PID.

(v) Is a regular 102-gon constructible? Why?

(vi) Is $\mathbf{Q}(\sqrt{7}, \sqrt{-2})$ a simple extension of $\mathbf{Q}$? Why?

2. [12%] Let $f(x)$ be an irreducible polynomial over $\mathbf{Q}$. Suppose that the degree of $f(X)$ is n. Let K be the splitting field of $f(X)$ over $\mathbf{Q}$. Show that

(i) n divides $[K : \mathbf{Q}]$;

(ii) $[K : \mathbf{Q}]$ divides $n!$.

3. [10%] Let K be the splitting field of $x^4 + x^2 - 6$ over $\mathbf{Q}$. Find the Galois group $G(K/\mathbf{Q})$.

4. [10%] Let p be a prime integer. Let F be a finite field of order p^4 and E be a finite field of order p^{24} such that $F \leq E$. Find the Galois group $G(E/F)$.

5. [12%] Let K be the splitting field of the 15-$\underline{\text{th}}$ cyclotomic polynomial $\Phi_{15}(x)$ over $\mathbf{Q}$. Describe the Galois group $G(K/\mathbf{Q})$ by the fundamental theorem of finitely-generated abelian groups.

6. [10%] Show that $\mathbf{C}$ is a finite normal extension of $\mathbf{R}$. Then find the Galois group $G(\mathbf{C}/\mathbf{R})$.

7. [10%] Suppose that E is a finite normal extension of $\mathbf{Q}$, and $E \leq \mathbf{C}$. Suppose that $E \cap \mathbf{R} \neq E$. Show that the order of the Galois group $G(E/\mathbf{Q})$ is even.

8. [12%] Let $f(x) = x^4 + x + 1 \in \mathbf{Z}_2[x]$, and α be a zero of $f(x)$. Show that the minimal polynomial of α^7 over $\mathbf{Z}_2$ is $x^4 + x^3 + 1$ (i.e., $\operatorname{irr}(\alpha^7, \mathbf{Z}_2) = x^4 + x^3 + 1$).