FINAL FOR ALGEBRA

Date: 2000, January 17, 9:10–11:00AM

Each of the following problems worth 10 points.

1.

(i) Give an example of a group of order 4 which is not cyclic.
(ii) Give an example of a infinite non-abelian group.
(iii) Give an example of a non-abelian solvable group.
(iv) Give an example of a non-commutative division ring.
(v) Give an example of an ideal I of a commutative ring R such that I is prime but not maximal.

2.

(i) What is the characteristic of the ring $\mathbf{Z}_6 \times \mathbf{Z}$? why?
(ii) What is the commutator subgroup of a simple non-abelian group? Why?
(iii) What is the order of the element $(12)(345)(12)$ in S_8? Why?

3. Suppose that H is a normal subgroup of a group G and K is a normal subgroup of H. Let a be an element in G.

(i) Show that $aKa^{-1} \subset H$.
(ii) Show that aKa^{-1} is a normal subgroup of H.

4.

(i) Find all prime number p such that $x+2$ is a factor of $x^4 + x^3 + x^2 - x + 1$ in $\mathbf{Z}_p[x]$.
(ii) Show that for p a prime, the polynomial $x^p + a$ in $\mathbf{Z}_p[x]$ is not irreducible for any $a \in \mathbf{Z}_p$.

5. Show that $\phi\colon \mathbf{C} \to M_2(\mathbf{R})$ given by

$$\phi(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

for $a, b \in \mathbf{R}$ gives an isomorphism of $\mathbf{C}$ with the subring $\phi[\mathbf{C}]$ of $M_2(\mathbf{R})$ where $M_2(\mathbf{R})$ is the ring of two by two matrices over $\mathbf{R}$.

6.

(i) Is $\mathbf{Q}[x]/\langle x^2 - 5x + 6\rangle$ a field? Why?
(i) Is $\mathbf{Q}[x]/\langle x^2 - 6x + 6\rangle$ a field? Why?

7. Let A and B be ideals of a ring R. The product AB of A and B is defined by

$$AB = \left\{ \sum_{i=1}^{n} a_i b_i \;\middle|\; a_i \in A,\ b_i \in B,\ n \in \mathbf{Z}^+ \right\}.$$

(i) Show that AB is an ideal of R.
(ii) Show that $AB \subseteq (A \cap B)$.

8. Let R be a commutative ring and N be an ideal of R. Define

$$\sqrt{N} = \{ a \mid a^n \in N \text{ for some } n \in \mathbf{Z}^+ \}.$$

(i) Show that $N \subseteq \sqrt{N}$ and $\sqrt{N}$ is an ideal of R.
(ii) Give an example of N such that $\sqrt{N} = N$.

(ii) Give an example of N such that $\sqrt{N} \neq N$.

9.

(i) Let K be a subgroup of index 2 of a group G. Suppose that $a \in G - K$ and $b \in G - K$ i.e., a, b are in G but not in K. Show that $ab \in K$.

(ii) Let G be a finite abelian group. Suppose that G has two distinct elements of order 2. Show that 4 divides $|G|$.

10. Let $\phi\colon \mathbf{R} \to \mathbf{R}$ be a nontrivial ring homomorphism.

(i) Show that $\phi(a) = a$ if $a \in \mathbf{Z}$.

(ii) Show that $\phi(a) = a$ if $a \in \mathbf{Q}$.

(iii) Show that $\phi[\mathbf{R}^+] \subseteq \mathbf{R}^+$ where $\mathbf{R}^+ = \{\, a \in \mathbf{R} \mid a > 0 \,\}$. (Hint: a square is positive.)

(iv) Show that $\phi(a) > \phi(b)$ if $a, b \in \mathbf{R}$ and $a > b$.

(v) Show that $\phi(a) = a$ for all $a \in \mathbf{R}$.