MIDTERM 1 FOR ADVANCED LINEAR ALGEBRA

Date: Wednesday, Nov 8, 2000
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No credit will be given for an answer without reasoning.

1.

(i) [5%] Give an example of a nondegenerate symmetric bilinear form of Witt index 1 on a two-dimensional real vector space.

(ii) [5%] Give an example of nonzero quadratic form on a two-dimensional real vector space.

2. Let $\mathbf{H}$ be the quaternion algebra over $\mathbf{R}$ and let $M_2(\mathbf{R})$ be the matrix algebra of two by two real matrices.

(i) [5%] Is $\mathbf{H}$ isomorphic to $M_2(\mathbf{R})$ as vector spaces over $\mathbf{R}$? Why or why not?

(ii) [5%] Is $\mathbf{H}$ isomorphic to $M_2(\mathbf{R})$ as $\mathbf{R}$-algebras? Why or why not?

3. Let $V = F^2$ be a two-dimensional vector space over a field F. Define two unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. It is obvious that $\{\mathbf{i}, \mathbf{j}\}$ is a basis for V. Define $f : V \times V \to F$ by

$$f((a_1, b_1), (a_2, b_2)) = 2b_1b_2$$

for $a_1, a_2, b_1, b_2 \in F$.

(i) [5%] What is the radical of V?

(ii) [5%] Suppose that the characteristic of F is not 3. Find the matrix presenting the form f with respect to the basis $\{9\mathbf{i}, 3\mathbf{j}\}$.

4. [10%] Find the Jordan canonical form of the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

5. Let $V = \mathbf{R}^2$ be a two-dimensional real vector space. Fix a basis $\{\mathbf{i}, \mathbf{j}\}$ for V where $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. Define

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2$$

for $x_1, x_2, y_1, y_2 \in \mathbf{R}$.

(i) [5%] Show that the linear transformation

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

where θ is a real number is an isometry of V.

(ii) [5%] Suppose that $\tau : V \to V$ is an isometry. Show that $a\tau$ for $a \in \mathbf{R}$ is an isometry if and only if $a = \pm 1$.

6. Let F be a field and V be a vector space over F. Suppose that f is a bilinear form on V.

(i) [5%] Show that if f is alternating, then f is skew-symmetric.

(ii) [5%] Show that if f is skew-symmetric and the characteristic of F is not 2, then f is alternating.

7. Let $V = \mathbf{C}^2$ be a two-dimensional vector space over $\mathbf{C}$. Define a symmetric bilinear form f on V by

$$f((x_1, x_2), (y_1, y_2)) = x_1 y_1 + x_2 y_2$$

for $x_1, x_2, y_1, y_2 \in \mathbf{C}$.

(i) [5%] Show that f is isotropic.

(ii) [5%] Show that V is a hyperbolic plane.

8. Let $V = \mathbf{R}^2$ be a two-dimensional vector space over a field $\mathbf{R}$ with a nondegenerate skew-symmetric bilinear form $\langle , \rangle$ defined by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 y_2 - y_1 x_2$$

for $x_1, x_2, y_1, y_2 \in \mathbf{R}$.

(i) [5%] Suppose that f is a linear functional on V. The Riesz representation theorem tells us that there is a unique element $x \in V$ such that $f = \phi_x$ where $\phi_x \in V^*$ is defined by $\phi_x(v) = \langle v, x \rangle$. Find x for the linear functional f defined by $f((x, y)) = 2x + 3y$ for $x, y \in \mathbf{R}$.

(ii) [5%] Let S be the subspace spanned by the vector $(1, 0)$. Is $V = S \perp S^{\perp}$? Why or why not?

9. [10%] Let $P_2 \subset \mathbf{R}[x]$ be the space of polynomials of degree less than or equal to 2. Define

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$$

for $f, g \in P_2$. We know that $\{1, x, x^2\}$ is a basis for P_2. Apply Gram-Schmidt orthogonalization process to $\{1, x, x^2\}$ to find an orthogonal basis for P_2.

10. Let ℓ^2 be the set of all real infinite sequences (a_n) such that $\sum_{n=1}^{\infty} |a_n|$ is finite. Define $(a_n) + (b_n) = (a_n + b_n)$ and $r(a_n) = (r a_n)$ for $(a_n), (b_n) \in \ell^2$ and $r \in \mathbf{R}$.

(i) [5%] Show that ℓ^2 is a vector space over $\mathbf{R}$.

(ii) [5%] Show that ℓ^2 is an inner product space under the inner product $\langle , \rangle$ defined by

$$\langle (a_n), (b_n) \rangle = \sum_{n=1}^{\infty} a_n b_n.$$