FINAL FOR ADVANCED CALCULUS

Date: Monday, Jun 17, 2002
Instructor: Shu-Yen Pan
No credit will be given for an answer without reasoning.

1. [10%]

(i) Give an example of a set $A \subset \mathbb{R}^n$ such that A is of measure zero but not of volume zero.
(ii) Give an example of a function f on $[-\pi, \pi]$ such that the Fourier series of f converges uniformly.
(iii) Give an example of a function f on $[-\pi, \pi]$ such that the Fourier series of f converges to f at every point x in $[-\pi, \pi]$ but not uniformly on $[-\pi, \pi]$.

2. [10%] Integrate the function $f(x, y, z) = x^2 + y^2 + z^2$ over the set $B = \{ (x, y, z) \mid x^2 + y^2 + z^2 \leq 1 \}$.

3. [10%] Suppose that $A \subset \mathbb{R}^n$ and A has zero volume. Suppose $f\colon A \to \mathbb{R}$ is a bounded function. Prove that f is integrable and $\int_A f = 0$.

4. [10%] Let $f_n(x) = \sum_{k=1}^{n} \frac{1}{2^k} \sin kx$ for $x \in \mathbb{R}$.

(i) Show that $\lim_{n\to\infty} f_n(x)$ exists for all x.
(ii) Show that the sequence converges uniformly on the whole real line.
(iii) Show that $\int_0^{2\pi} (\lim_{n\to\infty} f_n(t))\, dt = 0$

5. [10%] Let $\varphi_n(x) = \frac{1}{\sqrt{\pi}} \sin nx$. Check that $\{ \varphi_n(x) \mid n \in \mathbb{N} \}$ is an orthonormal family of functions on $[0, 2\pi]$.

6. [10%] Compute the Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ of the function $f(x) = x$ on the interval $[-\pi, \pi]$.

7. [10%] Let $\mathcal{V}$ be a Hilbert space and let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be a complete orthonormal set. Let $c_0, c_1, c_2, \ldots$ be complex numbers and suppose that $\sum_{k=0}^{\infty} |c_k|^2 < \infty$. Let $f_n = \sum_{k=0}^{n} c_k \varphi_k$.

(i) Show that $\|f_n - f_m\|^2 = \sum_{k=m+1}^{n} |c_k|^2$.
(ii) Show that f_n is a Cauchy sequence in $\mathcal{V}$.
(iii) Show that $\sum_{k=0}^{\infty} c_k \varphi_k$ is the Fourier series of some f.

8. [10%] Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist at $(0, 0)$ but are not equal.

9. [10%] Let $f(x) = \frac{\cos x}{x}$. Show that $\int_1^{\infty} f(x)\, dx$ converges but $\int_1^{\infty} |f(x)|\, dx$ does not converge.

10. [10%] Let $f\colon \mathbb{R} \to \mathbb{R}$ be of class C^1 and $u = f(x)$, $v = -y + xf(x)$. If $f'(x_0) \neq 0$, show that this transformation is invertible near (x_0, y_0) and the inverse has the form $x = f^{-1}(u)$, $y = -v + uf^{-1}(u)$.