Numerical Analysis Jan 7, 2011

1. Derive the Adams-Bashforth method of order 3 (13 %)

$$y_{n+1} = y_n + \frac{h}{12}\left[23\, y'_n - 16\, y'_{n-1} + 5\, y'_{n-2}\right]$$

2. Show that second order Runge–Kutta formulas of the form (13 %)

$$y_{n+1} = y_n + h\left[\gamma_1 f(x_n, y_n) + \gamma_2 f(x_n + \alpha h, y_n + \beta h f(x_n, y_n))\right]$$

should impose these conditions

$$\gamma_1 + \gamma_2 = 1, \qquad \gamma_2 \alpha = \frac{1}{2}, \qquad \gamma_2 \beta = \frac{1}{2}$$

3. Show that the Trapzoidal method (13 %)

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

is absolutely stable.

4. Implement the Gauss–Seidel iterative method as a MATLAB function. (13 %)

5. Show that the general residual correction method (13 %)

$$\begin{aligned} r^{(k)} &= b - Ax^{(k)} \\ N\hat{e}^{(k)} &= r^{(k)} \\ x^{(k+1)} &= x^{(k)} + \hat{e}^{(k)} \end{aligned}$$

is exactly the same as the method

$$Nx^{(k+1)} = b + Px^{(k)}.$$

Here, $A = N - P$.

6. Find the Cholesky factorization $A = LL^T$ for the matrix (13 %)

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 13 & 14 \\ 4 & 14 & 21 \end{pmatrix}$$

7. Suppose that $P_2(x) \in \Pi_2$ interpolates f at the three nodes $x_0 = x_1 - h$, x_1, and $x_2 = x_1 + h$. Show that (13 %)

$$P'(x_1) = \frac{f(x_1 + h) - f(x_1 - h)}{2h}$$

and

$$f'(x_1) = P_2'(x_1) - \frac{h^2}{6} f'''(c_2)$$

with $x_1 - h \le c_2 \le x_1 + h$. Note that Π_2 is the set of polynomials of degree ≤ 2

8. Work out the 3-node Gaussian numerical integration formula (13 %)

$$\int_{-1}^{1} f(x)\, dx \approx \omega_1 f(x_1) + \omega_2 f(x_2) + \omega_3 f(x_3)$$

by finding the nodes x_i and the weights ω_i so that the formula is exact for polynomials of degree less then or equal to 5. Hint: Legendre polynomial $P_3(x) = 5x^3 - 3x$.